Strategic Interaction between Hedge Funds and Prime Brokers

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Very preliminary version, please do not quote

Abstract

We develop a framework of strategic interaction between prime brokers and hedge funds. The hedge fund optimally determines its cash holdings and the fraction of shorted securities. The prime broker optimally determines its cash holdings, the margin rates, and the rehypothecation rate. The lending rate is determined at the equilibrium. Optimal decisions are obtained when the hedge fund and the prime broker maximize their expected return on equity. To do so, we describe how the evolution of the market return affects the equity of the hedge fund and may force it to delever or even default. As the eventual default of the hedge fund would severely affect the prime broker's performance, the broker tries to mitigate the risk induced by the fund by fixing the margin rates (or haircuts) it imposes to the hedge fund. We then explore the interaction between the hedge fund and the prime broker decisions by calibrating and solving our model for realistic parametrizations. We find that the interaction between the hedge fund and the prime broker may give rise to some undesirable implications such as an increase in overall risk and/or leverage.

Keywords: Hedge Fund, Prime Broker, Leverage, Balance sheet

JEL: G2, G23, G24

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1 Introduction

According to Preqin (2015), hedge funds added over \$355 billion in additional capital in 2014, due to both the performance of hedge funds and inflows from investors. Much of this increase in industry assets has come from institutional investors, with 44% of fund managers reporting an increase in the amount of capital coming from these investors over the course of 2014. As reported by EVestment, in summer 2014 the total assets in the hedge fund industry reached \$3 trillion.

Leverage is a crucial component of the asset management strategy of hedge funds. According to the report of Financial Service Authority (Financial Services Authority, 2012), within the asset management industry, the hedge fund sector has the highest values of leverage. The fragility and vulnerability of hedge funds are associated with their leverage.[1] Leverage may be direct through borrowing, e.g., from prime brokers, or indirect through off-balance sheet positions, e.g. derivative instruments. Prime brokers are the most important service providers for hedge funds, not only through financing leverage, but also through other services such as risk management, execution, or custody. Prime brokerage has two important organization characteristics. On the one hand, the prime broker service is very concentrated. The top three (twelve) brokers service 41.06% (75.03%) of hedge funds. On the other hand, hedge funds usually rely on a limited number of prime brokers. According to Finadium (2013), half of the funds still have only one prime broker. In addition, the recent trend suggests that newly established hedge funds are more dependent on their prime brokers. In 2015 less than 3% of hedge funds have been launched with the help of three or more prime brokers, down from 11.4% in 2014 and 14.5% in 2008. More than 70% of the launches used just one bank, up from 61% in 2014 and 55% in 2012 (Eurekahedge). In fact, the hedge fund and its prime broker are mutually risky to each other. On one side, large losses or default by the hedge fund can generate large losses for its prime broker. On the other side, the prime broker can withdraw capital from hedge fund by increasing lending rates or margin rates, exposing the hedge fund to funding liquidity risk. The interaction between hedge fund and prime

[1] As hedge funds usually do not disclose their positions, there is no accurate information about their leverage levels. Estimates suggest that the leverage of hedge funds is widely dispersed across strategies.

broker decisions has led Agarwal, Mullally, and Naik (2015) to raise the question whether hedge funds alone are responsible for risk propagation due to their leverage, or their prime brokers also play a role by poor decisions in margin financing provision.

To address this question, we develop a framework that describes the interaction between the prime broker's decisions and the hedge fund's financing strategy. We consider one representative hedge fund and one representative prime broker, which are both risk neutral and maximize their expected return on equity for the next period. The hedge fund obtains financing for its long and short positions from the prime broker through its margin account. Most of these loans are collateralized. The major share of the prime broker's income comes from collateralized cash lending and securities lending to facilitate short-selling. The model explains how the demand and supply of leverage by the hedge fund and the prime broker determine the margin rates (haircuts), the lending rate, and the rehypothecation rate. The hedge fund optimally determines the amount of cash holdings and its long and short positions. The prime broker optimally chooses its cash holdings, the margin rates, and the rehypothecation rate. The lending rate is determined in the equilibrium by adding to the risk-free rate a risk premium that reflects the expected loss of the prime broker in case of a default of the hedge fund. The contribution of the paper to the previous research is that the model pins down all these variables jointly. To our knowledge, this is also the first paper to explicitly model the balance sheet of a prime broker.

The role of prime brokers in increasing the funding liquidity risk of hedge funds may be potentially amplified by financial regulation. The new requirements lead to changes in balance sheet strategy of prime brokers. Basel III regulation may reduce the leverage prime brokers offer to hedge funds and the extent of that may vary across hedge funds. The liquidity coverage ratio and the net stable funding ratio force prime brokers to better match the maturity of their assets and liabilities, i.e., their own funding and the financing offered to hedge funds. Prime brokers need to rely on secured financing for longer terms. Hence, it could lead to higher borrowing rates for prime brokers and decrease the spread between haircuts for prime brokers and for hedge funds.

The remainder of the paper is organized as follows. Section 2 reviews the literature. Section 3 presents the model for the optimal decision of the hedge fund. Section 4 describes the framework for the prime broker. Section 5 details how the equilibrium lending rate is determined. Section 6 discusses the results. Section 7 draws conclusions.

2 Literature Review

The paper contributes to the literature that explores the performance of hedge funds related to their service providers. Olaru (2006) builds a model of the relationship between prime brokers and hedge funds. Klaus and Rzepkowski (2009) show that, during financial distress of prime brokers, there is a decline in hedge fund performance and that the hedge funds that rely on multiple prime brokers have higher returns. Mirabile (2015) finds that funds that have chosen the most popular domicile and leading service providers have lower performance than those who made other choices. Goldie (2012) explores the influence of prime brokers on involvement of hedge funds in merger and acquisition activities. Cumming, Dai, and Johan (2013) explore hedge funds which domicile is Delaware. They find that hedge funds domiciled there do not outrun or perform more poorly than other funds in terms of returns. Hespeler and Witt (2014) examine the influence of prime brokers on the illiquidity of the hedge funds. Gerasimova (2014) and Chung and Kang (2016) find strong comovement of the returns of hedge funds dealing with the same prime broker.

The paper also belongs to the large strand of research that explores leverage of hedge funds. Duffie, Wang, and Wang (2009), Dai and Sundaresan (2010), Lan, Wang, and Yang (2013), Buraschi, Kosowski, and Sritrakul (2014) set up theoretical models of optimal leverage in the presence of management fees, insolvency losses, funding costs, etc. There is a significant number of papers that examine empirically the level of leverage. Ang, Gorovyy, and van Inwege (2011) analyze leverage of hedge funds using exclusive data from funds of funds. They conclude that changes in hedge fund leverage tend to be more predictable by economy-wide factors than by fund-specific characteristics. McGuire and Tsatsaronis (2008) estimate using beta exposures that the hedge fund leverage lies between one and ten times of assets under management. Farnsworth (2014) examines whether hedge funds successfully adjust their leverage to time the returns on the assets.

The paper is related to research on funding liquidity risk and leverage of intermediaries. Adrian and Shin (2010), Adrian and Shin (2014) and Adrian, Etula, and Muir (2014) analyze leverage and risk of broker-dealers and their impact on the other agents. Dudley and Nimalendran (2011) explore the link between funding risk and hedge fund contagion. More precisely, they find that an increase in margin requirements (e.g., a decrease in funding liquidity) leads to increased correlation in hedge fund returns. Liu and Mello (2011) examine hedge fund capital structure and show that funding liquidity risk explains hedge funds' cash holdings. In their model, optimal cash holdings reflect a trade-off between reducing liquidation costs and increasing return by holding risky, non-cash assets.

3 Financing Decision of the Hedge Fund

In this section, we determine the optimal strategy of the hedge fund regarding its financing. We assume that the investment strategy is related to the market portfolio in a very simple way and consider the problem of the financing of this strategy. We start by describing how the strategy is financed. Then we define the structure and the dynamics of the balance sheet. Finally, we explain how the hedge fund determines its optimal financing strategy.

3.1 Generating Leverage in a Hedge Fund

There are different instruments available to hedge funds to generate leverage. Three main sources of leverage can be used: (1) Hedge funds can explicitly borrow external funds from banks; (2) They can implicitly borrow funds through margin brokerage account and/or shorting securities; (3) Last, they can use derivatives instruments (swaps, futures, forwards) or structured products.

The second source of financing (prime brokerage) is a key source of financing for hedge funds because it allows them to raise funds and to short sell, both through collateralization. It is relatively easy to obtain and cheap relative to borrowing from banks. However, its financing terms may change with short notice depending on the prime broker willingness to lend. It will play an important role in the subsequent analysis.

A first way to get financing from the prime broker is to *buy on margin*. A hedge fund wants to buy securities on margin, by borrowing money from its prime broker. The number of shares bought of security j is $n^+_{j,t}$ at price $p_{j,t}$. The prime broker agrees to lend $n^+_{j,t}l_{j,t}$ provided the hedge fund invests $n^+_{j,t}\tilde{\mu}^+_{j,t}$ of its own equity. Therefore, $n^+_{j,t}l_{j,t}$ is the credit line and $n^+_{j,t}\tilde{\mu}^+_{j,t}$ is the margin. The prime broker holds the securities bought with the credit line as collateral. The collateral initially represents a fraction $(p_{j,t}/l_{j,t})$ of the prime broker lending. Summing over all the long positions in securities, the long position financed through buying on margin is $L^H_t = \sum_j n^+_{j,t}p_{j,t}$ and the margin account on long positions is $M^+_t = \sum_j n^+_{j,t}\tilde{\mu}^+_{j,t}$.[2]

The hedge fund can also *sell* assets *short*. The number of shares of security j sold is $n^-_{j,t}$ at price $p_{j,t}$. The fund is requested by the prime broker to deposit $n^-_{j,t}\tilde{\mu}^-_{j,t}$ of its own equity on the margin account. The hedge fund then borrows the securities, sells them on the market, and receives the proceeds in cash. In principle, except for short-only funds, the cash proceeds may be invested in long positions in other securities. The short proceeds can be held in cash or invested in securities. In both cases, the proceeds are held by the prime broker as collateral and therefore cannot be used as collateral to finance other transactions. The collateral initially represents a fraction $(\tilde{\mu}^-_{j,t} + p_{j,t})/p_{j,t}$ of the short position. We assume that the collateral is composed of the cash proceeds of the short sale. Summing over all the securities shorted, the short position is $S^H_t = \sum_j n^-_{j,t}p_{j,t}$ and the margin account on short positions is $M^-_t = \sum_j n^-_{j,t}\tilde{\mu}^-_{j,t}$.

The motivations of both mechanisms are different: buying on margin aims at generating leverage, whereas short selling aims at constructing a long/short portfolio. Yet both mechanisms result in leverage. As the hedge fund has both long and short positions, two definitions of leverage are useful to describe its balance sheet. The gross leverage is based on the sum of the long and short positions: $Lev_{G,t} = (L^H_t + S^H_t)/N^H_t$. The net leverage is based on the difference of the long and short positions: $Lev_{N,t} = (L^H_t - S^H_t)/N^H_t$. The extend of leverage is driven by the initial margin rates for long positions $\mu^+_{j,t} = \tilde{\mu}^+_{j,t}/p_{j,t}$

[2]To avoid some ambiguity, the superscripts "H" and "P" stand for hedge fund and prime broker.

and for short positions $\mu_{j,t}^{-} = \tilde{\mu}_{j,t}^{-}/p_{j,t}$. In the following, we will generically denote by $\mu_{L,t}$ and $\mu_{S,t}$ the initial margin rate for long and short positions.

The prime broker does not allow the hedge fund to build excessive leverage because it would increase its own counterparty risk. The limit is that the margin account should not exceed the fund's capital, denoted by N_t^H. Therefore, the total margin M_t^H should satisfy:

$$M_t^H = M_t^+ + M_t^- = \sum_j (n_{j,t}^+ \tilde{\mu}_{j,t}^+ + n_{j,t}^- \tilde{\mu}_{j,t}^-) \leq N_t^H. \tag{1}$$

The difference $C_t^H = N_t^H - M_t^H$ is the free (or unencumbered) cash.

We use the following simplified balance sheet for hedge fund in our subsequent analysis:

Schema 1: Simplified balance sheet of a hedge fund

Assets		**Liabilities and Equity**	
Free cash	(C_t^H)	Debt	(D_t^H)
Cash proceeds	(P_t^H)	Short securities	(S_t^H)
Long securities	(L_t^H)	Equity	(N_t^H)

In the framework, at the beginning of the period, we have the following relations hold between the various items of the balance sheet:

$$\begin{aligned} N_t^H &= M_t^+ + M_t^- + C_t^H, \\ M_t^+ &= L_t^H - D_t^H, \\ M_t^- &= P_t^H - S_t^H. \end{aligned}$$

As time goes by, it may be that the value of the long position $(L_{t+1}^H = (1 + r_{L,t+1}) L_t^H)$ goes down, whereas the debt only increases in proportion of the predetermined lending rate $(D_{t+1}^H = (1+r_{D,t+1}) D_t^H)$. In such case, the value of the margin account also goes down $(M_{t+1}^+ = L_{t+1}^H - D_{t+1}^H < M_t^+)$. When the value of the margin account goes below a given fraction of the long position, the hedge fund has to put more cash on the margin account or to delever. The limit is defined by the maintenance margin rate, denoted by $m_{L,t}$, such that $M^+_{t+1} \geq m_{L,t} L^H_{t+1}$ for all dates. Similarly, it may happen that the value of the short positions ($S^H_{t+1} = (1 + r_{S,t+1})S^H_t$) goes up, whereas the cash proceeds deposited on the prime broker account only increase with the risk-free rate ($P^H_{t+1} = (1 + r_{F,t+1})P^H_t$). In such a case, the value of the margin account also goes down ($M^-_{t+1} = P^H_{t+1} - S^H_{t+1} < M^-_t$). When the value of the margin account goes below a given fraction of the short position, the hedge fund has to put more cash on the margin account or to delever. The limit is defined by the maintenance margin rate, denoted by $m_{S,t}$, such that $M^-_{t+1} \geq m_{S,t} S^H_{t+1}$ for all dates. Therefore, the hedge fund has to ensure that its margin account satisfies:

$$
\begin{aligned}
M^+_t &= \max(L^H_t - D^H_t, m_{L,t} L^H_{t+1}), \\
M^-_t &= \max(P^H_t - S^H_t, m_{S,t} S^H_{t+1}).
\end{aligned}
$$

To simplify the analysis, we assume that the hedge fund can rely on portfolio margining.[3] The hedge fund does not have to satisfy the margin limit on the long positions and short positions separately, but it has to satisfy the limit on the combined portfolio:

$$
M^H_{t+1} = M^+_{t+1} + M^-_{t+1} = \max(L^H_{t+1} - D^H_{t+1} + P^H_{t+1} - S^H_{t+1}, m_{L,t} L^H_{t+1} + m_{S,t} S^H_{t+1}), \quad \text{for all } t.
$$

In words, if the hedge fund has excess margin on the short position, the margin can be used to finance the margin call on the long position, and vice versa.

3.2 Structure of the Balance Sheet

Now, we define the decision variables of the hedge regarding its financing structure. The margin account is set by the hedge fund to be a fraction α_t of its own equity, $M^H_t = \alpha_t N^H_t$, so that the remaining equity, $C^H_t = (1 - \alpha_t)N^H_t$, is held in unencumbered cash. Unencumbered cash would be used by the hedge fund in case of margin calls or if the prime broker increases the margin rate. If the hedge fund reaches its maximum leverage

[3]Portfolio margin is a risk-based margin system that defines margin rates based on the overall risk of the portfolio. As a consequence, the margin rates are usually much lower than with Regulation T (approximately 15% for equity).

($\alpha_t = 1$), an increase in the margin rate would force it to delever by liquidating some of its positions. Therefore, the unencumbered cash plays the role of a buffer. The fraction α_t is called margin multiplier.

The fund posts a fraction $\alpha_t \gamma_t$ of its own equity on the margin account to buy on margin. Given a margin rate $\mu_{L,t}$, the fund borrows $[\alpha_t \gamma_t (1 - \mu_{L,t})/\mu_{L,t}] N_t^H$ and invests $[\alpha_t \gamma_t / \mu_{L,t}] N_t^H$ into new securities. The fund also uses a fraction $\alpha_t(1 - \gamma_t)$ of its equity on the margin account to sell short. It borrows and sells $[\alpha_t(1 - \gamma_t)/\mu_{S,t}] N_t^H$ of securities and holds the proceeds in cash.

As we can see in Schema 2, the balance sheet of the hedge fund is determined by two parameters once the equity is scaled to N_t^H: the initial margin rates ($\mu_{L,t}$ for long positions and $\mu_{S,t}$ for short positions). The structure of the balance sheet is then optimized by two main decisions of the hedge fund: the margin multiplier α_t and the fraction of capital γ_t used to buy on margin (with $1 - \gamma_t$ used for short selling). The parameter γ_t depends on the investment strategy of the hedge fund: γ_t is typically close to 1 for a long-only hedge fund and smaller than 0.5 for a dedicated short bias hedge fund.

Schema 2: Parametrized balance sheet of a hedge fund at date t

	Assets		**Liabilities and Equity**		
(r_F)	Free cash	$(1 - \alpha_t) N_t^H$	Margin debt	$\alpha_t \gamma_t \frac{1-\mu_{L,t}}{\mu_{L,t}} N_t^H$	(r_D)
(r_F)	Cash proceeds	$\alpha_t (1 - \gamma_t) \frac{1+\mu_{S,t}}{\mu_{S,t}} N_t^H$	Short securities	$\alpha_t (1 - \gamma_t) \frac{1}{\mu_{S,t}} N_t^H$	(r_S)
(r_L)	Long securities	$\alpha_t \gamma_t \frac{1}{\mu_{L,t}} N_t^H$	Equity	N_t^H	(r_{NH})
			- Free cash	$(1 - \alpha_t) N_t^H$	
			- Long account	$\alpha_t \gamma_t N_t^H$	
			- Short account	$\alpha_t (1 - \gamma_t) N_t^H$	

To investigate the impact of market shocks on the hedge fund return on equity, we assume that the return on the long and short positions of the fund is related to the market return as follows:

$$r_{L,t+1} = \beta_L \, r_{M,t+1}, \tag{2}$$

$$r_{S,t+1} = \beta_S \, r_{M,t+1}. \tag{3}$$

In doing so, we allow the long and short positions to have contrasting exposures to the market shock, in particular in the downturn.[4] For the empirical evaluation, we assume for convenience that the market return $r_{M,t+1}$ has a log-normal distribution, i.e., $\log(1 + r_{M,t+1}) \sim N(\mu_M, \sigma_M)$.

3.3 Dynamics of the Expected Return on Equity

We now turn to the dynamics of the expected return on equity. Shocks hitting the hedge fund (market shocks or changes in the prime broker policy or in investors decisions) may require some adjustments from the hedge fund such as margin calls, fire sales, or even default. For this purpose, we need some additional assumptions. First, we assume that β_L and β_S, i.e., the investment strategy of the hedge fund, do not change in the short term, so that they are not instantaneously affected by a market shock.[5] Second, decisions made by the prime broker and the investors generate two types of funding liquidity risks for the hedge fund: (1) the prime broker can reduce the access of the hedge fund to liquidity by increasing its initial margin or maintenance margin (margin funding risk); (2) the investors can withdraw funds (redemption risk) (Klaus and Rzepkowski, 2009).

Assets under management (AUM) are defined as: $N_t^H = NAV_t \ Sh_t$, where NAV_t is the net asset value of a fund's share and Sh_t is the number of shares outstanding at date t. The dynamics of the AUM is therefore given by:

$$N_{t+1}^H = (1 + r_{NH,t+1})NAV_t(1 + \varphi_{t+1})Sh_t = (1 + r_{NH,t+1})(1 + \varphi_{t+1})N_t^H, \qquad (4)$$

where $r_{NH,t+1}$ is the return on the NAV between t and $t + 1$ and φ_{t+1} is the fraction of new shares issued between t and $t + 1$ net of the shares redeemed. When $\varphi_{t+1} > 0$, the number of shares increases (net inflows); when $\varphi_{t+1} < 0$, the number of shares decreases (net outflows). The issuance rate φ_{t+1} is likely to be related to the recent evolution of the fund performance. We assume that the issuance rate is $\varphi_{t+1} = 0$ for the moment.

[4] By selecting $\gamma_t = \mu_{L,t}\beta_S/(\mu_{L,t}\beta_S + \mu_{S,t}\beta_L)$, the hedge fund could neutralize its exposure to market shocks. However, as the strategies do not generate extra return ("alpha"), we do not expect the hedge fund to be market neutral.

[5] This assumption means that, if the fund has to sell assets to satisfy margin calls, it will delever uniformly over all asset classes.

Next, we expect that the prime broker will be more reluctant to lend and will increase its initial margin rates ($\mu_{L,t}$ and $\mu_{S,t}$) and the maintenance margin rates ($m_{L,t}$ and $m_{S,t}$) when the market or the hedge fund have bad performances. For instance, Dai and Sundaresan (2010) assume that the prime broker is more restrictive on its funding when the fund risky portfolio incurs large losses. In case of a change in the margin rate, the hedge fund would have to adjust its portfolio so that it satisfies the new limits set by the prime broker. In practice, margin rates are likely to change in case of extreme stress. The CGFS (2010) reports such values before and after the start of the 2008 crisis.

When the market conditions deteriorate, there is a sequence of actions that the hedge fund can take to reduce its leverage and avoid default.[6] We now make some assumptions on the sequence. In particular, we assume that there are costs associated with the various actions and that the hedge fund rationally starts with the less expensive ones. In case of a market fall, we expect the hedge fund to incur losses on its risky portfolio. In case of a margin call, the fund will first use its free cash, as it is available at no cost. Then, if the fund does not have sufficient cash, it will liquidate its short positions. In a market fall, buying back securities can be done at a relatively low cost because prices are going down. Finally, if cash obtained from the short positions is not enough, the fund will liquidate its long position. The liquidation is done at a relatively higher cost because stock prices are already under pressure. We denote by θ and ϕ ($\theta < \phi$) the cost paid by the hedge fund to liquidate its short and long positions.[7] Eventually, if liquidating the long position is insufficient, the hedge fund cannot repay its debt and defaults.

[6] The decision to liquidate some long or short positions can be directly undertaken by the prime broker.

[7] θ and ϕ could incorporate the transaction fees paid by the hedge fund to the prime broker for the execution of the trades. However, they should also potentially incorporate the price impact of the trades, which would be more favorable for closing a short position than a long position.

3.3.1 Normal time

In normal time, the net asset value of the hedge fund at the end of period t is decomposed as follows:

$$\begin{aligned} N^H_{t+1} &= (1+r_{NH,t+1})N^H_t \\ &= (1+r_{F,t+1})(C^H_t+P^H_t)+(1+r_{L,t+1})L^H_t-(1+r_{D,t+1})D^H_t-(1+r_{S,t+1})S^H_t, \end{aligned}$$

so that the return on equity depends on the initial structure of the balance sheet as:

$$\begin{aligned} 1+r_{NH,t+1} &= (1+r_{F,t+1})\frac{\alpha_t(1-\gamma_t)+\mu_{S,t}(1-\alpha_t\gamma_t)}{\mu_{S,t}}+(1+r_{L,t+1})\frac{\alpha_t\gamma_t}{\mu_{L,t}} \\ &\quad -(1+r_{D,t+1})\frac{\alpha_t\gamma_t(1-\mu_{L,t})}{\mu_{L,t}}-(1+r_{S,t+1})\frac{\alpha_t(1-\gamma_t)}{\mu_{S,t}}. \end{aligned}$$

Using relations (2) and (3), we can express the return on equity of the fund as a function of the market conditions as follows:

$$\begin{aligned} r_{NH,t+1} &= r_{M,t+1}\left(\beta_L\frac{\alpha_t\gamma_t}{\mu_{L,t}}-\beta_S\frac{\alpha_t(1-\gamma_t)}{\mu_{S,t}}\right)+r_{F,t+1}\frac{\alpha_t(1-\gamma_t)+\mu_{S,t}(1-\alpha_t\gamma_t)}{\mu_{S,t}} \\ &\quad -r_{D,t+1}\frac{\alpha_t\gamma_t(1-\mu_{L,t})}{\mu_{L,t}}. \end{aligned}$$

Margin calls happen when the margin account (M^H_{t+1}) of the fund is below the maintenance ratio ($m_{L,t}$ and $m_{S,t}$) applied to the long and short portfolios (margin limits):[8]

$$M^H_{t+1}=N^H_{t+1}-C^H_{t+1}<m_{L,t}\ L^H_{t+1}+m_{S,t}\ S^H_{t+1}.$$

This expression gives:

$$\begin{aligned} &(1+r_{L,t+1})(1-m_{L,t})\frac{\alpha_t\gamma_t}{\mu_{L,t}}-(1+r_{S,t+1})(1+m_{S,t})\frac{\alpha_t(1-\gamma_t)}{\mu_{S,t}} \\ &\qquad <(1+r_{D,t+1})\frac{\alpha_t\gamma_t(1-\mu_{L,t})}{\mu_{L,t}}-(1+r_{F,t+1})\alpha_t(1-\gamma_t)\frac{1+\mu_{S,t}}{\mu_{S,t}}, \end{aligned}$$

[8] In Federal Reserve Board Regulation T, the initial margin rate for equity purchases and short sales is set equal to $\mu_{L,t}=\mu_{S,t}=50\%$ since 1974, so that the maximum level of both long and short exposures is equal to 2. The maintenance margin rates are set equal to $m_{L,t}=m_{S,t}=25\%$.

or, in terms of market return,:

$$r_{M,t+1}\left((1-m_{L,t})\frac{\alpha_t\gamma_t}{\mu_{L,t}}\beta_L-(1+m_{S,t})\frac{\alpha_t(1-\gamma_t)}{\mu_{S,t}}\beta_S\right)<(1+r_{D,t+1})\frac{\alpha_t\gamma_t(1-\mu_{L,t})}{\mu_{L,t}}$$
$$-(1+r_{F,t+1})\alpha_t(1-\gamma_t)\frac{1+\mu_{S,t}}{\mu_{S,t}}-(1-m_{L,t})\frac{\alpha_t\gamma_t}{\mu_{L,t}}+(1+m_{S,t})\frac{\alpha_t(1-\gamma_t)}{\mu_{S,t}},$$

which we rewrite as: $r_{M,t+1}<\bar{r}^{(MC)}_{M,t+1}$.

When the performance of the market portfolio is below the threshold $\bar{r}^{(MC)}_{M,t+1}$, the hedge fund receives a margin call and has to refund its margin account. Three solutions are available: (1) either the fund has sufficient free cash to repay part of its debt, (2) or it has to buy back part of its short positions, (3) or it has to sell part of its long positions to satisfy the maintenance ratio. We consider the three cases in turn.

3.3.2 Using free cash only

In the case of a margin call (if $r_{M,t+1}<\bar{r}^{(MC)}_{M,t+1}$), the hedge fund starts by using its free cash. It reduces its debt using ΔC^H_{t+1} of cash to a level such that the margin limit is satisfied, i.e.:

$$M^H_{t+1}=M^+_{t+1}+M^-_{t+1}=m_{L,t}(1+r_{L,t+1})L^H_t+m_{S,t}(1+r_{S,t+1})S^H_t,$$

with

$$\begin{aligned} M^+_{t+1} &= (1+r_{L,t+1})L^H_t-(1+r_{D,t+1})D^H_t+\Delta C^H_{t+1} \\ M^-_{t+1} &= (1+r_{F,t+1})P^H_t-(1+r_{S,t+1})S^H_t. \end{aligned}$$

We find:

$$\begin{aligned} \Delta C^H_{t+1} = N^H_t\Bigg[&-(1+r_{L,t+1})(1-m_{L,t})\frac{\alpha_t\gamma_t}{\mu_{L,t}}+(1+r_{S,t+1})(1+m_{S,t})\frac{\alpha_t(1-\gamma_t)}{\mu_{S,t}} \\ &+(1+r_{D,t+1})\frac{\alpha_t\gamma_t(1-\mu_{L,t})}{\mu_{L,t}}-(1+r_{F,t+1})\alpha_t(1-\gamma_t)\frac{1+\mu_{S,t}}{\mu_{S,t}}\Bigg]. \end{aligned}$$

We deduce that value of the new debt as:

$$\begin{aligned} D^H_{t+1} &= (1+r_{D,t+1})D^H_t - \Delta C^H_{t+1} = N^H_t \Bigg[(1+r_{L,t+1})(1-m_{L,t})\frac{\alpha_t\gamma_t}{\mu_{L,t}} \\ &\quad -(1+r_{S,t+1})(1+m_{S,t})\frac{\alpha_t(1-\gamma_t)}{\mu_{S,t}} + (1+r_{F,t+1})\alpha_t(1-\gamma_t)\frac{1+\mu_{S,t}}{\mu_{S,t}} \Bigg]. \end{aligned}$$

The return on equity is not directly affected by the reallocation of free cash. The use of all the free cash is insufficient to satisfy the margin limit as soon as $(1+r_{F,t+1})C^H_t < \Delta C^H_{t+1}$, which gives the following condition:

$$\begin{aligned} &(1+r_{L,t+1})(1-m_{L,t})\frac{\alpha_t\gamma_t}{\mu_{L,t}} - (1+r_{S,t+1})(1+m_{S,t})\frac{\alpha_t(1-\gamma_t)}{\mu_{S,t}} < \\ &\qquad (1+r_{D,t+1})\frac{\alpha_t\gamma_t(1-\mu_{L,t})}{\mu_{L,t}} - (1+r_{F,t+1})\frac{\alpha_t(1-\gamma_t)+\mu_{S,t}(1-\alpha_t\gamma_t)}{\mu_{S,t}}. \end{aligned}$$

In terms of market return, we find:

$$\begin{aligned} r_{M,t+1}&\left((1-m_{L,t})\frac{\alpha_t\gamma_t}{\mu_{L,t}}\beta_L - (1+m_{S,t})\frac{\alpha_t(1-\gamma_t)}{\mu_{S,t}}\beta_S\right) < (1+r_{D,t+1})\frac{\alpha_t\gamma_t(1-\mu_{L,t})}{\mu_{L,t}} \\ &-(1+r_{F,t+1})\frac{\alpha_t(1-\gamma_t)+\mu_{S,t}(1-\alpha_t\gamma_t)}{\mu_{S,t}} - (1-m_{L,t})\frac{\alpha_t\gamma_t}{\mu_{L,t}} + (1+m_{S,t})\frac{\alpha_t(1-\gamma_t)}{\mu_{S,t}}, \end{aligned}$$

which we rewrite as: $r_{M,t+1} < \bar{r}^{(C)}_{M,t+1}$.

3.3.3 Buying back short positions

If the fund has to liquidate some of its risky positions ($r_{M,t+1} < \bar{r}^{(C)}_{M,t+1}$), we assume that it starts by liquidating short positions because it is less costly in a market fall.[9] The fund has already exhausted its free cash, so that $\Delta C^H_{t+1} = (1+r_{F,t+1})C^H_t$ and $C^H_{t+1} = 0$. Now the fund buys back ΔS^H_{t+1} of its short positions to a level such that the margin limit is satisfied, i.e.:

$$M^+_{t+1} + M^-_{t+1} = m_{L,t}(1+r_{L,t+1})L^H_t + m_{S,t}((1+r_{S,t+1})S^H_t - \Delta S^H_{t+1}),$$

[9] The strategy of liquidating short positions to finance the margin call on long positions is possible because we have assumed portfolio margining.

with

$$
\begin{aligned}
M_{t+1}^{+} &= (1+r_{L,t+1})L_t^H - (1+r_{D,t+1})D_t^H + (1+r_{F,t+1})C_t^H \\
M_{t+1}^{-} &= (1+r_{F,t+1})P_t^H - (1+r_{S,t+1})S_t^H - \theta \Delta S_{t+1}^H,
\end{aligned}
$$

where θ denotes the cost of liquidating the short position. As the short position is reduced by ΔS_{t+1}^H but the cash proceeds by $(1+\theta)\Delta S_{t+1}^H$, the short margin also decreases by $\theta \Delta S_{t+1}^H$. We find:

$$
\begin{aligned}
\Delta S_{t+1}^H &= \frac{N_t^H}{m_{S,t}-\theta}\Bigg[-(1+r_{L,t+1})(1-m_{L,t})\frac{\alpha_t\gamma_t}{\mu_{L,t}} + (1+r_{S,t+1})(1+m_{S,t})\frac{\alpha_t(1-\gamma_t)}{\mu_{S,t}} \\
&+(1+r_{D,t+1})\frac{\alpha_t\gamma_t(1-\mu_{L,t})}{\mu_{L,t}} - (1+r_{F,t+1})\frac{\alpha_t(1-\gamma_t)+\mu_{S,t}(1-\alpha_t\gamma_t)}{\mu_{S,t}}\Bigg].
\end{aligned}
$$

The value of the debt is not affected. Instead, the cash proceeds are reduced as:

$$
\begin{aligned}
P_{t+1}^H &= (1+r_{F,t+1})P_t^H - (1+\theta)\Delta S_{t+1}^H = \frac{N_t^H}{m_{S,t}-\theta}\Bigg[(1+\theta)(1+r_{L,t+1})(1-m_{L,t})\frac{\alpha_t\gamma_t}{\mu_{L,t}} \\
&-(1+\theta)(1+r_{S,t+1})(1+m_{S,t})\frac{\alpha_t(1-\gamma_t)}{\mu_{S,t}} - (1+\theta)(1+r_{D,t+1})\frac{\alpha_t\gamma_t(1-\mu_{L,t})}{\mu_{L,t}} \\
&+(1+r_{F,t+1})\left((1+\theta)(1-\alpha_t) + (1+m_{S,t})\alpha_t(1-\gamma_t)\frac{1+\mu_{S,t}}{\mu_{S,t}}\right)\Bigg].
\end{aligned}
$$

The return on equity is:

$$
\begin{aligned}
1+r_{NH,t+1} &= \frac{1}{m_{S,t}-\theta}\Bigg[(1+r_{L,t+1})\frac{\alpha_t\gamma_t}{\mu_{L,t}}(m_{S,t}-\theta m_{L,t}) - (1+r_{S,t+1})\frac{\alpha_t(1-\gamma_t)}{\mu_{S,t}}m_{S,t}(1+\theta) \\
&+ (1+r_{F,t+1})\frac{\alpha_t(1-\gamma_t)+\mu_{S,t}(1-\alpha_t\gamma_t)}{\mu_{S,t}}m_{S,t} - (1+r_{D,t+1})\frac{\alpha_t\gamma_t(1-\mu_{L,t})}{\mu_{L,t}}m_{S,t}\Bigg].
\end{aligned}
$$

The liquidation of the short positions is insufficient to satisfy the margin limit as soon as $(1+r_{S,t+1})S_t^H < \Delta S_{t+1}^H$, which gives the following condition:

$$
\begin{aligned}
&(1+r_{L,t+1})(1-m_{L,t})\frac{\alpha_t\gamma_t}{\mu_{L,t}} - (1+r_{S,t+1})(1+\theta)\frac{\alpha_t(1-\gamma_t)}{\mu_{S,t}} < \\
&\quad (1+r_{D,t+1})\frac{\alpha_t\gamma_t(1-\mu_{L,t})}{\mu_{L,t}} - (1+r_{F,t+1})\frac{\alpha_t(1-\gamma_t)+\mu_{S,t}(1-\alpha_t\gamma_t)}{\mu_{S,t}}.
\end{aligned}
$$

In terms of market return, we find:

$$r_{M,t+1}\left((1-m_{L,t})\frac{\alpha_t\gamma_t}{\mu_{L,t}}\beta_L-(1+\theta)\frac{\alpha_t(1-\gamma_t)}{\mu_{S,t}}\beta_S\right)<(1+r_{D,t+1})\frac{\alpha_t\gamma_t(1-\mu_{L,t})}{\mu_{L,t}}$$
$$-(1+r_{F,t+1})\frac{\alpha_t(1-\gamma_t)+\mu_{S,t}(1-\alpha_t\gamma_t)}{\mu_{S,t}}-(1-m_{L,t})\frac{\alpha_t\gamma_t}{\mu_{L,t}}+(1+\theta)\frac{\alpha_t(1-\gamma_t)}{\mu_{S,t}},$$

which we rewrite as: $r_{M,t+1}<\bar{r}^{(S)}_{M,t+1}$.

3.3.4 Selling long positions

We now consider the case where the fund has to liquidate some of its long positions ($r_{M,t+1}<\bar{r}^{(S)}_{M,t+1}$). The fund has already exhausted its free cash, so that $\Delta C^H_{t+1}=(1+r_{F,t+1})C^H_t$ and $C^H_{t+1}=0$, and its short positions, so that $\Delta S^H_{t+1}=(1+r_{S,t+1})S^H_t$ and $S^H_{t+1}=0$. Now the fund sells ΔL^H_{t+1} of its long positions to a level such that the margin limit is satisfied, i.e.:

$$M^+_{t+1}+M^-_{t+1}=m_{L,t}((1+r_{L,t+1})L^H_t-\Delta L^H_{t+1}),$$

with

$$\begin{aligned}M^+_{t+1}&=(1+r_{L,t+1})L^H_t-(1+r_{D,t+1})D^H_t+(1+r_{F,t+1})C^H_t-\phi\Delta L^H_{t+1}\\ M^-_{t+1}&=(1+r_{F,t+1})P^H_t-(1+\theta)(1+r_{S,t+1})S^H_t,\end{aligned}$$

where ϕ denotes the cost of liquidating long positions ($\phi>\theta$). We find:

$$\begin{aligned}\Delta L^H_{t+1}&=\frac{N^H_t}{m_{L,t}-\phi}\Bigg[-(1+r_{L,t+1})(1-m_{L,t})\frac{\alpha_t\gamma_t}{\mu_{L,t}}+(1+r_{S,t+1})(1+\theta)\frac{\alpha_t(1-\gamma_t)}{\mu_{S,t}}\\&\quad+(1+r_{D,t+1})\frac{\alpha_t\gamma_t(1-\mu_{L,t})}{\mu_{L,t}}-(1+r_{F,t+1})\frac{\alpha_t(1-\gamma_t)+\mu_{S,t}(1-\alpha_t\gamma_t)}{\mu_{S,t}}\Bigg].\end{aligned}$$

We assume that, when the hedge fund has exhausted all its short positions, the excess cash proceeds (if any) are used to reduce the remaining debt. The value of the new debt is then given by:

$$\begin{aligned} D_{t+1}^{H} &= (1+r_{D,t+1})D_t^H - (1+r_{F,t+1})C_t^H - (1-\phi)\Delta L_{t+1}^H \\ &\quad -[(1+r_{F,t+1})P_t^H - (1+r_{S,t+1})(1+\theta)S_t^H] \\ &= \frac{N_t^H(1-m_{L,t})}{m_{L,t}-\phi}\Bigg[(1+r_{L,t+1})(1-\phi)\frac{\alpha_t\gamma_t}{\mu_{L,t}} - (1+r_{S,t+1})(1+\theta)\frac{\alpha_t(1-\gamma_t)}{\mu_{S,t}} \\ &\quad -(1+r_{D,t+1})\frac{\alpha_t\gamma_t(1-\mu_{L,t})}{\mu_{L,t}} + (1+r_{F,t+1})\frac{\alpha_t(1-\gamma_t)+\mu_{S,t}(1-\alpha_t\gamma_t)}{\mu_{S,t}}\Bigg]. \end{aligned}$$

The return on equity is:

$$\begin{aligned} 1+r_{NH,t+1} &= \frac{m_{L,t}}{m_{L,t}-\phi}\Bigg[(1+r_{L,t+1})\frac{\alpha_t\gamma_t}{\mu_{L,t}}(1-\phi) - (1+r_{S,t+1})\frac{\alpha_t(1-\gamma_t)}{\mu_{S,t}}(1+\theta) \\ &+ (1+r_{F,t+1})\frac{\alpha_t(1-\gamma_t)+\mu_{S,t}(1-\alpha_t\gamma_t)}{\mu_{S,t}} - (1+r_{D,t+1})\frac{\alpha_t\gamma_t(1-\mu_{L,t})}{\mu_{L,t}}\Bigg]. \end{aligned}$$

The liquidation of the long position is insufficient to satisfy the margin limit as soon as $(1+r_{L,t+1})L_t^H < \Delta L_{t+1}^H$, which gives the following condition:

$$\begin{aligned} &(1+r_{L,t+1})(1-\phi)\frac{\alpha_t\gamma_t}{\mu_{L,t}} - (1+r_{S,t+1})(1+\theta)\frac{\alpha_t(1-\gamma_t)}{\mu_{S,t}} < \\ &\qquad (1+r_{D,t+1})\frac{\alpha_t\gamma_t(1-\mu_{L,t})}{\mu_{L,t}} - (1+r_{F,t+1})\frac{\alpha_t(1-\gamma_t)+\mu_{S,t}(1-\alpha_t\gamma_t)}{\mu_{S,t}}. \end{aligned}$$

In terms of market return, we find:

$$\begin{aligned} r_{M,t+1}\left((1-\phi)\frac{\alpha_t\gamma_t}{\mu_{L,t}}\beta_L - (1+\theta)\frac{\alpha_t(1-\gamma_t)}{\mu_{S,t}}\beta_S\right) &< (1+r_{D,t+1})\frac{\alpha_t\gamma_t(1-\mu_{L,t})}{\mu_{L,t}} \\ -(1+r_{F,t+1})\frac{\alpha_t(1-\gamma_t)+\mu_{S,t}(1-\alpha_t\gamma_t)}{\mu_{S,t}} &- (1-\phi)\frac{\alpha_t\gamma_t}{\mu_{L,t}} + (1+\theta)\frac{\alpha_t(1-\gamma_t)}{\mu_{S,t}}, \end{aligned}$$

which we rewrite as: $r_{M,t+1} < \bar{r}_{M,t+1}^{(DE)}$. It corresponds to the default threshold.[10]

There is a default when the hedge fund has to sell all its securities ($\Delta L_{t+1}^H = (1+r_{L,t+1})L_t^H$) and the debt is still positive, so that it cannot repay all its debt. In such a

[10] We note that the default condition $r_{NH,t+1} \leq -1$ exactly corresponds to the condition $r_{M,t+1} < \bar{r}_{M,t+1}^{(DE)}$ when $m_{L,t} = m_{S,t}$.

case, the value of the debt that cannot be repaid is given by:

$$\begin{aligned} Loss^H_{D,t+1} &= (1+r_{D,t+1})D^H_t - (1+r_{F,t+1})(C^H_t + P^H_t) \\ &\quad -(1+r_{L,t+1})(1-\phi)L^H_t + (1+r_{S,t+1})(1+\theta)S^H_t. \end{aligned}$$

Figure 1 illustrates the value of the hedge fund's return on equity as a function of the market return, for the baseline parametrization we adopt in Section 6.[11] We observe that, when the market return falls below -12.4%, the maintenance rate for long positions is reached and the hedge fund has to put more cash on its margin account. As it holds some free cash, it starts by using it. When $r_{M,t+1}$ is below -13.1%, the free cash is exhausted and the hedge fund starts liquidating some of its short positions. When $r_{M,t+1}$ is below -14.8%, the short positions are also exhausted and the hedge fund starts liquidating some of its long positions. Finally, when $r_{M,t+1}$ is below -27%, the hedge fund has liquidated all its long positions and defaults.

We note that this evaluation is performed independently from the decision of the prime broker (i.e., with μ_L, μ_S, m_L, and m_S exogenously given). In Section 6, we analyze the interaction between the hedge fund and the prime broker with more details.

3.4 Optimal Strategy

Depending on the performance of the securities portfolio, the hedge fund faces five possible situations: normal time, use of free cash, buy backs, fire sales, or default. The expected

[11] In particular, we assume that the margin rates that apply to the hedge fund are $\mu_L = \mu_S = 30\%$ and $m_L = m_S = 20\%$ and the risk-free is set equal to 1.5% per year. With this parametrization, the hedge fund optimally uses $\alpha^* = 98.3\%$ of its equity for financing its risky positions and holds 1.7% in free cash. In addition, $\gamma^* = 92\%$ of its financing equity is used for long positions. Last, the equilibrium lending rate is found to be equal to $r^*_D = 1.58\%$ per year.

return of the fund is therefore:

$$\begin{aligned}
E_t[r_{NH,t+1}] &= E_t\left[r_{NH,t+1} \mid r_{M,t+1} > \overline{r}^{(MC)}_{M,t+1}\right](1 - F(\log(1+\overline{r}^{(MC)}_{M,t+1}))) \\
&+ E_t\left[r_{NH,t+1} \mid r_{M,t+1} \in [\overline{r}^{(C)}_{M,t+1}; \overline{r}^{(MC)}_{M,t+1}]\right](F(\log(1+\overline{r}^{(MC)}_{M,t+1})) - F(\log(1+\overline{r}^{(C)}_{M,t+1}))) \\
&+ E_t\left[r_{NH,t+1} \mid r_{M,t+1} \in [\overline{r}^{(SS)}_{M,t+1}; \overline{r}^{(C)}_{M,t+1}]\right](F(\log(1+\overline{r}^{(C)}_{M,t+1})) - F(\log(1+\overline{r}^{(S)}_{M,t+1}))) \\
&+ E_t\left[r_{NH,t+1} \mid r_{M,t+1} \in [\overline{r}^{(DE)}_{M,t+1}; \overline{r}^{(S)}_{M,t+1}]\right](F(\log(1+\overline{r}^{(S)}_{M,t+1})) - F(\log(1+\overline{r}^{(DE)}_{M,t+1}))) \\
&- F(\log(1+\overline{r}^{(DE)}_{M,t+1})),
\end{aligned}$$

where $F(\log(1+\overline{r}^{(\vartheta_L)}_{M,t+1})) = \Phi(\log(1+\overline{r}^{(\vartheta_L)}_{M,t+1}); \mu_M; \sigma_M)$. We notice that:

$$\begin{aligned}
&E_t[r_{M,t+1} \mid r_{M,t+1} \in [\overline{r}^{(\vartheta_L)}_{M,t+1}; \overline{r}^{(\vartheta_U)}_{M,t+1}]](F(\log(1+\overline{r}^{(\vartheta_U)}_{M,t+1})) - F(\log(1+\overline{r}^{(\vartheta_L)}_{M,t+1}))) = \\
&\qquad \exp(\mu_M + \tfrac{1}{2}\sigma_M^2)(G(\log(1+\overline{r}^{(\vartheta_U)}_{M,t+1})) - G(\log(1+\overline{r}^{(\vartheta_L)}_{M,t+1}))),
\end{aligned}$$

where $G(\log(1+\overline{r}^{(\vartheta_L)}_{M,t+1})) = \Phi(\log(1+\overline{r}^{(\vartheta_L)}_{M,t+1}); \mu_M + \sigma_M^2; \sigma_M)$.

The initial margin rates ($\mu_{L,t}$ and $\mu_{S,t}$), the maintenance rates ($m_{L,t}$ and $m_{S,t}$), the risk-free rate ($r_{F,t+1}$), the lending rate ($r_{D,t+1}$), and the distribution of the market return (μ_M and σ_M) are given to the hedge fund. We also assume that the strategy (β_L and β_S) is fixed in the short run. Then the hedge fund determines its optimal strategy, i.e., the optimal margin multiplier (α_t) and the fraction of the risky portfolio invested in long positions (γ_t). The optimal parameters are found by maximizing the expected return on equity:

$$\max_{\{\alpha_t,\gamma_t\}} E_t[r_{NH,t+1}],$$

with the restrictions $0 \leq \alpha_t, \gamma_t \leq 1$.

Figures 2 to 9 show the optimal decisions of the hedge fund as a function of the various exogenous parameters. For each key parameter, we show the margin multiplier α_t^*, the fraction of long positions γ_t^*, the expected return on equity $r_{NH,t+1}$, the probability of default of the hedge fund $F(\log(1+\overline{r}^{(DE)}_{M,t+1}))$ implied by the hedge fund's strategy, the gross leverage $(L_t^H + S_t^H)/N_t^H$ and the equilibrium lending rate $r_{D,t+1}$.

As the expected market return is low, the hedge fund is "market neutral", which implies that the probability of default is very low and the expected return on equity is also low (Figure 2). As the expected market return increases, the fund increases its long positions (γ^* increases to 1). For a low market volatility, the hedge fund has a net long position but no free cash (Figure 3). With high volatility, the hedge fund is long only to benefit from the higher sensitivity of the long position to the market return. At the same time, it also increases its free cash because the likelihood of margin calls is higher. It does not prevent the probability of default and the lending rate to increase significantly (up to 2.5% and 2.2% per year, respectively, with $\sigma_M = 28\%$ per year).

Next figures illustrate how the hedge fund tries to benefit from the exposure of its risky positions to the market return. When the sensitivity β_L increases, the hedge fund switches from a long/short strategy to a long-only strategy, while keeping some free cash to cover the margin call (Figure 4). In this situation the probability of default and the lending rates increase significantly (to 4% and 2.75% per year, respectively for $\beta_L = 1.5$). If the hedge fund manages to find a short strategy with a low β_S, it is able to generate a high return on equity with a long/short strategy (Figure 5). In contrast, when β_S is higher (and closer to β_L), the hedge fund cannot generate a high return with a long/short strategy. Therefore, for high β_S, the hedge fund invests in the long-only strategy, with some free cash, even though the return on equity is low.

The impact of the margin rates on the optimal decisions of the hedge fund can be observed in next figures, in which we assume $\mu_L = \mu_S$ and $m_L = m_S$. With a low initial margin rate, the hedge fund can generate high leverage (Figure 6). Therefore, it invests in a (more profitable) long/short strategy so that it increases the expected return on equity. With high $\mu_L = \mu_S$, the hedge fund is limited in its ability to leverage its strategy, which reduces the probability of default, but also the return on equity. A low value of $m_L = m_S$ (for a fixed $\mu_L = \mu_S$) allows the hedge fund to leverage more, as the maintenance margin is less binding (Figure 7). Therefore, it can generate a higher return on equity at the expense of a higher probability of default and lending rate.

Finally, next figures show the impact of the liquidation costs (ϕ and θ) on the hedge fund's financing decisions. As expected, a higher cost of liquidating long positions (ϕ)

implies a lower investment in the long strategy (Figure 8). Inversely an increase in the cost of liquidating short positions (θ) is accompanied by a larger weight in long positions (Figure 9). In both cases, the expected return on equity is reduced. As θ increases, the probability of default is also higher because the hedge fund's is less diversified and more affected by a market downturn.

4 Financing Decision of the Prime Broker

This section describes the optimal strategy of the prime broker regarding its lending to the hedge fund and its borrowing from other banks or money market funds. We present the structure and the dynamics of the prime broker's balance sheet. The major share of prime broker's income comes from trading commissions, collateralized cash lending, and stock or bond lending to facilitate short-selling. The securities lending service gives rise to dependency of the hedge fund towards its prime broker. Through its margin requirements and collateral risk management, the prime broker influences the amount of leverage employed by the hedge fund. The prime broker tends to increase hedge fund collateral requirements and mandate haircuts in the event of extended stressful market conditions, thus inducing forced deleveraging of risky positions.[12]

The hedge fund obtains financing for its long and short positions from its prime broker. Most of these loans are secured ones. The prime broker usually has a right to rehypothecate (part of) the collateral.[13] It optimally chooses the amount of free cash, the margin rates, the lending fees, and the fraction of collateral that is rehypothecated.

[12]The hedge funds relying on the service of the most affected brokers during the last crisis such as Bear Stearns or Lehman Brothers were as a result more likely to face higher funding liquidity risk and therefore to obtain lower returns. Aragon and Strahan (2012) find that hedge funds that used Lehman Brothers as their prime broker could not trade after the bankruptcy, and the probability of failure for these funds was twice higher than for similar funds who used other prime brokers.

[13]The rehypothecation rate influences how the hedge fund might be exposed to the bankruptcy of its prime broker. Hence, higher rehypothecation rate should lead to lower lending rate (from prime broker to hedge fund) and/or lower margin rates. The regulation of rehypothecation differs between countries. In the U.S., Federal Reserve Board Regulation T and SEC Rule 15c3-3 limit the amount of a client's assets which a prime broker may rehypothecate to the equivalent of 140% of the client's liability to the prime broker. In other markets, there are no such limits.

4.1 Structure of the Balance Sheet

When a security is borrowed from a broker and sold short, the hedge fund receives cash proceeds from the sale. The short proceeds are deposited on the broker account and are paid an interest. Certain prime brokerage arrangements allow the borrower to reinvest the proceeds to purchase additional securities long. Prime brokerage is limited in the level of leverage it can provide.

In general, the balance sheet of a prime broker is composed of the items described in Schema 3.

Schema 3: Balance Sheet of a Prime Broker

Assets	**Liabilities and Equity**
Cash and Cash Equivalents	Unsecured Short-term Borrowings
Cash and Securities Segregated	Collateralized Financings:
Collateralized Agreements:	- Repo
- Reverse Repo	- Securities Loaned
- Securities Borrowed	Payables:
Receivables:	- to Brokers, Dealers and Clearing Org.
- from Brokers, Dealers and Clearing Org.	- to Customers and Counterparties
- from Customers and Counterparties	Financial Instruments Sold
Financial Instruments Owned	Equity

We now describe our simplified version of the balance sheet (Schema 4). We only consider the trades of the prime broker in relation to the financing of the hedge fund. There are two sequences of such trades. First, the hedge fund buys on margin. The prime broker provides $[\alpha_t \gamma_t (1 - \mu_{L,t})/\mu_{L,t}] N_t^H$ of cash to the hedge fund, which buys $[\alpha_t \gamma_t / \mu_{L,t}] N_t^H$ of securities on the market. The securities are posted as a collateral to secure the buying on margin. The prime broker repledges a fraction ρ_t of the collateral to another investor, which in turn provides a loan of $[\rho_t \alpha_t \gamma_t (1 - \mu_{L,t}^P)/\mu_{L,t}] N_t^H$, where $\mu_{L,t}^P$ is the margin rate for long position used for interbroker collateralized transactions.[14] This loan finances part of the initial loan done by the prime broker to the hedge fund. The remaining part, $[\alpha_t\gamma_t((1-\mu_{L,t})-\rho_t(1-\mu^P_{L,t}))/\mu_{L,t}]N^H_t$, is borrowed on the (unsecured) interbank market without collateral. The interest rates used for this sequence are the following: the lending rate for the loan (with collateral) of the prime broker to the hedge fund is denoted by $r_{D,t+1}$; the borrowing rate of the prime broker is denoted by $r_{C,t+1}$ (with repledged collateral); finally, the unsecured interbank rate for the remaining part of the borrowing is denoted by $r_{I,t+1}$.

Second, the hedge fund enters a short sale for $[\alpha_t(1-\gamma_t)/\mu_{S,t}]N^H_t$. The prime broker needs to borrow the securities from another investor. To secure the securities borrowing, it deposits $[\alpha_t(1-\gamma_t)(1-\mu^P_{S,t})/\mu_{S,t}]N^H_t$ of cash, where $\mu^P_{S,t}$ is the margin rate for short position used for interbroker collateralized transactions. Then, the securities are loaned to the hedge fund, which deposits $[\alpha_t(1-\gamma_t)(1-\mu_{S,t})/\mu_{S,t}]N^H_t$ of cash proceeds on the prime broker account. As the cash proceeds deposited by the hedge fund is larger than the cash used by the prime broker to borrow the securities (assuming $\mu_{S,t} > \mu^P_{S,t}$), the prime broker has some excess of cash $[\alpha_t(1-\gamma_t)(\mu_{S,t}-\mu^P_{S,t})/\mu_{S,t}]N^H_t$, which is invested on the (unsecured) interbank market. Cash proceeds are assumed to be remunerated at the risk-free rate ($r_{F,t+1}$), whereas the cash invested on the interbank market is remunerated at interest rate $r_{I,t+1}$.

To summarize, the prime broker generates a profit through two channels: (1) The difference between the lending rate ($r_{D,t+1}$) and the borrowing rate ($r_{C,t+1}$) for the hedge fund's long position. This profit is partly offset by the need of the prime broker to borrow on the unsecured market the part of the securities that is not repledged. (2) The difference between the cost of financing of the short position $[\alpha_t(1-\gamma_t)(\mu_{S,t}-\mu^P_{S,t})/\mu_{S,t}]N^H_t$ at rate $r_{F,t+1}$) and the remuneration of the excess cash invested on the interbank market (at rate $r_{I,t+1}$).

[14]The other investor may be another branch of the bank the prime broker belongs to. To avoid unnecessary complications, we assume that they are different entities.

Schema 4: Parametrized Balance Sheet of a Prime Broker at date t

	Assets		Liabilities and Equity		
(r_I)	**Unsecured cash**	$\alpha_t(1-\gamma_t)\frac{\mu_{S,t}-\mu^P_{S,t}}{\mu_{S,t}}N^H_t + (1-\alpha^P_t)N^P_t$	**Unsecured borrowing**	$\alpha_t\gamma_t\frac{1-\mu_{L,t}-\rho_t(1-\mu^P_{L,t})}{\mu_{L,t}}N^H_t$	(r_I)
	Collateralized Agreements:		**Collateralized Financings:**		
(r_F)	Sec. borrowed	$\alpha_t(1-\gamma_t)\frac{1+\mu^P_{S,t}}{\mu_{S,t}}N^H_t$	Sec. loaned	$\alpha_t(1-\gamma_t)\frac{1+\mu_{S,t}}{\mu_{S,t}}N^H_t$	(r_F)
	Receivables:		**Payables:**		
(r_D)	from HF	$\alpha_t\gamma_t\frac{1-\mu_{L,t}}{\mu_{L,t}}N^H_t$	to investor	$\rho_t\alpha_t\gamma_t\frac{1-\mu^P_{L,t}}{\mu_{L,t}}N^H_t$	(r_C)
(r_M)	**Sec. portfolio**	$A^P_t = \alpha^P_t\ N^P_t$	**Equity**	N^P_t	(r_{NP})
			- Free cash	$(1-\alpha^P_t)N^P_t$	
			- Other	$\alpha^P_t\ N^P_t$	

Some remarks are of importance: First, we assume that both the unsecured borrowings and cash are remunerated at the interbank rate, $r_{I,t+1}$. Therefore, in principle, the prime broker would not hold unnecessary cash. However, free cash could allow the prime broker to survive in case of a default of the hedge fund. Therefore, we assume that the prime broker will optimally hold a fraction $(1-\alpha^P_t)$ of total equity in free cash. Second, to simplify the analysis, as financial instrument owned and sold but not yet purchased that are not related to the hedge fund activities are out of scope of our paper, we do not consider their modeling in the sequel. We determine the amount of securities portfolio (owned net of sold but not yet purchased) by assuming that it is financed by equity, i.e., $A^P_t = \alpha^P_t\ N^P_t$. Third, we assume a minimum equity rule such as Debt $\leq \vartheta$ Equity, where ϑ is a measure of regulatory maximum leverage ratio. Assuming that the prime broker holds the minimum equity imposed by the rule, the initial equity is given by:[15]

$$N^P_t = \frac{1}{\vartheta}\left[\frac{\alpha_t\gamma_t}{\mu_{L,t}}[1-\mu_{L,t}-\rho_t(1-\mu^P_{L,t})] + \alpha_t(1-\gamma_t)\frac{1+\mu_{S,t}}{\mu_{S,t}} + \rho_t\alpha_t\gamma_t\frac{1-\mu^P_{L,t}}{\mu_{L,t}}\right]N^H_t,$$

[15]Assume a minimum capital ratio equal to 8% of the assets would give a value of $\vartheta = 11.5$.

which depends on the decisions of both the hedge fund (α_t and γ_t) and the prime broker (ρ_t and the margin rates $\mu_{L,t}$ and $\mu_{S,t}$). From this value, we deduce the amount of free cash and the amount invested in the securities portfolio.

4.2 Dynamics of the Expected Return on Equity

4.2.1 Normal time

We now turn to the dynamics of the expected return on equity. In normal time, the equity of the prime broker at the end of period t is decomposed as follows:

$$
\begin{aligned}
N^P_{t+1} &= (1+r_{N^P,t+1})N^P_t \\
&= (1+r_{D,t+1})D^H_t - (1+r_{C,t+1})D^P_t - (1+r_{I,t+1})(D^H_t - D^P_t) \\
&+ (r_{I,t+1} - r_{F,t+1})(P^H_t - P^P_t) + (1+r_{I,t+1})C^P_t + (1+r_{M,t+1})A^P_t \\
&= N^H_t \Bigg[(1+r_{D,t+1})\alpha_t\gamma_t \frac{1-\mu_{L,t}}{\mu_{L,t}} - (1+r_{C,t+1})\rho_t\alpha_t\gamma_t \frac{1-\mu^P_{L,t}}{\mu_{L,t}} \\
&- (1+r_{I,t+1})\alpha_t\gamma_t \frac{1-\mu_{L,t}-\rho_t(1-\mu^P_{L,t})}{\mu_{L,t}} + (r_{I,t+1} - r_{F,t+1})\alpha_t(1-\gamma_t)\frac{\mu_{S,t}-\mu^P_{S,t}}{\mu_{S,t}}\Bigg] \\
&+ N^P_t\left[(1+r_{I,t+1})(1-\alpha^P_t) + (1+r_{M,t+1})\alpha^P_t\right],
\end{aligned}
$$

where $D^P_t = [\rho_t\alpha_t\gamma_t(1-\mu^P_{L,t})/\mu_{L,t}]N^H_t$ denotes the collateralized debt of the prime broker payable to the other investor, based on repledged securities (interest rate $r_{C,t+1}$). Therefore, the return on equity is:

$$
\begin{aligned}
1 + r_{NP,t+1} &= \frac{N^H_t}{N^P_t}\Bigg[(1+r_{D,t+1})\alpha_t\gamma_t \frac{1-\mu_{L,t}}{\mu_{L,t}} - (1+r_{C,t+1})\rho_t\alpha_t\gamma_t \frac{1-\mu^P_{L,t}}{\mu_{L,t}} \\
&- (1+r_{I,t+1})\alpha_t\gamma_t \frac{1-\mu_{L,t}-\rho_t(1-\mu^P_{L,t})}{\mu_{L,t}} + (r_{I,t+1} - r_{F,t+1})\alpha_t(1-\gamma_t)\frac{\mu_{S,t}-\mu^P_{S,t}}{\mu_{S,t}}\Bigg] \\
&+ \left[(1+r_{I,t+1})(1-\alpha^P_t) + (1+r_{M,t+1})\alpha^P_t\right].
\end{aligned}
$$

It is worth noting that the return on equity of the prime broker in normal time only depends on market performances through the securities owned and sold but not yet purchased, which are not directly related to the prime brokerage business. Essentially, the prime broker makes money on the difference between the cost of financing of the hedge fund and its own cost of financing. In case of a market fall, the margin account of the hedge fund may be insufficient to cover its risky positions and the prime broker may request more funds (margin call). The actions taken by the hedge fund to satisfy the margin call will in turn have an impact on the performance of the prime broker. We consider the impact in the next sections.

4.2.2 The hedge fund uses free cash

When the hedge fund only relies on free cash to cover the margin call, it reduces its debt to the prime broker by ΔC^H_{t+1}. The prime broker reduces its own unsecured debt on the interbank market by the same amount. If the unsecured debt turns out to be fully repaid, the prime broker increases its free cash. Therefore, the equity of the prime broker at $t+1$ is:

$$
\begin{aligned}
N^P_{t+1} &= [(1+r_{D,t+1})D^H_t - \Delta C^H_{t+1}] - (1+r_{C,t+1})D^P_t - [(1+r_{I,t+1})(D^H_t - D^P_t) - \Delta C^H_{t+1}] \\
&+ (r_{I,t+1} - r_{F,t+1})(P^H_t - P^P_t) + (1+r_{I,t+1})C^P_t + (1+r_{M,t+1})A^P_t.
\end{aligned}
$$

We observe that the equity and the return on equity of the prime broker at $t+1$ are not directly affected when the hedge fund uses free cash to cover the margin call.

4.2.3 The hedge fund buys back short positions

When market return is below $\bar{r}^{(C)}_{M,t+1}$, the hedge fund has already exhausted all its free cash. The fund buys back some of its short positions for an amount equal to ΔS^H_{t+1}. Therefore, the hedge fund uses an amount $(1+\theta)\Delta S^H_{t+1}$ of its cash proceeds to buy back short positions, where the additional cost θ is due to the liquidation. The prime broker reduces its short positions by the same amount. Assuming a cost of liquidation of θ' ($\theta' < \theta$), it pays $(1+\theta')\Delta S^H_{t+1}$ of cash proceeds to the other investor and reduces its short position by the same amount ΔS^H_{t+1}. The consequence for the prime broker's balance sheet is that the collateralized loan is reduced by $(1+\theta)\Delta S^H_{t+1}$, whereas the collateralized financing is reduced by $(1+\theta')\Delta S^H_{t+1}$.

The equity of the prime broker at $t+1$ is:

$$\begin{aligned} N_{t+1}^P &= [(1+r_{D,t+1})D_t^H - (1+r_{F,t+1})C_t^H] - (1+r_{C,t+1})D_t^P \\ &- [(1+r_{I,t+1})(D_t^H - D_t^P) - (1+r_{F,t+1})C_t^H] + (1+r_{I,t+1})(P_t^H - P_t^P) \\ &- [(1+r_{F,t+1})P_t^H - (1+\theta)\Delta S_{t+1}^H] + [(1+r_{F,t+1})P_t^P - (1+\theta')\Delta S_{t+1}^H] \\ &+ [(1+r_{I,t+1})C_t^P + (1+r_{M,t+1})A_t^P], \end{aligned}$$

so that the return on equity is increased by $(\theta-\theta')\Delta S_{t+1}^H / N_t^P$:

$$\begin{aligned} 1+r_{NP,t+1} &= \frac{N_t^H}{N_t^P}\Bigg[-(1+r_{C,t+1})\rho_t\alpha_t\gamma_t\frac{1-\mu_{L,t}^P}{\mu_{L,t}} - (1+r_{I,t+1})\alpha_t\gamma_t\frac{1-\mu_{L,t}-\rho_t(1-\mu_{L,t}^P)}{\mu_{L,t}} \\ &\quad +(r_{I,t+1}-r_{F,t+1})\alpha_t(1-\gamma_t)\frac{\mu_{S,t}-\mu_{S,t}^P}{\mu_{S,t}}\Bigg] \\ &+ \frac{N_t^H}{(m_{S,t}-\theta)N_t^P}\Bigg[(m_{S,t}-\theta')(1+r_{D,t+1})\frac{\alpha_t\gamma_t(1-\mu_{L,t})}{\mu_{L,t}} \\ &\quad -(\theta-\theta')(1+r_{L,t+1})(1-m_{L,t})\frac{\alpha_t\gamma_t}{\mu_{L,t}} + (\theta-\theta')(1+r_{S,t+1})(1+m_{S,t})\frac{\alpha_t(1-\gamma_t)}{\mu_{S,t}} \\ &\quad -(\theta-\theta')(1+r_{F,t+1})\frac{\alpha_t(1-\gamma_t)+\mu_{S,t}(1-\alpha_t\gamma_t)}{\mu_{S,t}}\Bigg] \\ &+ [(1+r_{I,t+1})(1-\alpha_t^P) + (1+r_{M,t+1})\alpha_t^P]. \end{aligned}$$

In this case, the performance of the prime broker is affected by the liquidation of the short positions by the hedge fund because it has to partly liquidate its own short positions. We note that the performance of the prime broker benefits from the difference in the costs of liquidation (provided $\theta' < \theta$), but is also worsened by the bad performance of the securities portfolio.

4.2.4 The hedge fund sells long positions

When market return is below $\bar{r}_{M,t+1}^{(S)}$, the hedge fund has already exhausted its short positions. The excess cash proceeds (if any) has been used to reduce the remaining debt. Similarly, the prime broker has liquidated its short positions, which were the counterpart of the hedge fund's short positions, and used the excess cash proceeds (if any) to reduce the unsecured debt.

Now the hedge fund liquidates some of its long positions for an amount ΔL_{t+1}^H. Therefore, it repays part of its debt $(1-\phi)\Delta L_{t+1}^H$ to the prime broker, where the additional cost ϕ is due to the fire sale. The prime broker reduces its own unsecured debt on the interbank market by the same amount. If the unsecured debt is fully repaid, the prime broker increases its free cash. In doing so, the hedge fund also reduces the collateral available to the prime broker for rehypothecation. Therefore, the prime broker also has to reduce its own collateralized loan. We assume that the rehypothecation rate ρ_t is kept constant. This assumption implies that the value of the collateral repledged by the prime broker is now $\rho_t[(1+r_{L,t+1})L_t^H - \Delta L_{t+1}^H]$. We impose that the prime broker satisfies a maintenance margin rate equal to $m_{L,t}^P$, so that its margin account should be at least equal to $m_{L,t}^P \rho_t[(1+r_{L,t+1})L_t^H - \Delta L_{t+1}^H]$. In other words, its secured debt $(1+r_{C,t+1})D_t^P$ should be such that:

$$\rho_t[(1+r_{L,t+1})L_t^H - \Delta L_{t+1}^H] - (1+r_{C,t+1})D_t^P \geq m_{L,t}^P \rho_t[(1+r_{L,t+1})L_t^H - \Delta L_{t+1}^H].$$

This condition is equivalent to the following threshold for the return on hedge fund's long securities:

$$1+r_{L,t+1} \geq \frac{(1+r_{C,t+1})D_t^P - m_{L,t}^P \rho_t \Delta L_{t+1}^H}{(1-m_{L,t}^P)\rho_t L_t^H} = (1+r_{C,t+1})\frac{1-\mu_{L,t}^P}{1-m_{L,t}^P} + \frac{\Delta L_{t+1}^H}{L_t^H},$$

or, in terms of market return:

$$r_{M,t+1} \geq \frac{1}{\beta_L}\left[(1+r_{C,t+1})\frac{1-\mu_{L,t}^P}{1-m_{L,t}^P} + \frac{\Delta L_{t+1}^H}{L_t^H} - 1\right].$$

Above this threshold, the prime broker just reduces its secured debt by $\rho_t \Delta L_{t+1}^H$ and increases its unsecured debt by the same amount because it has less collateral available for rehypothecation. Therefore, the expressions for the equity and return on equity of the prime broker are not altered by the liquidation of long positions by the hedge fund.

In contrast, below the threshold, the prime broker has to reduce its secured debt further to satisfy the maintenance rate:

$$D^P_{t+1} = (1 - m^P_{L,t})\rho_t[(1 + r_{L,t+1})L^H_t - \Delta L^H_{t+1}].$$

Consequently, it also increases its unsecured borrowing by: $[(1+r_{C,t+1})D^P_t - D^P_{t+1}](1+\phi')$, where ϕ' denotes the adjustment cost for increasing the unsecured debt with short notice. In this case, the equity of the prime broker at $t+1$ is:

$$\begin{aligned}
N^P_{t+1} &= \Big[(1 + r_{D,t+1})D^H_t - (1 + r_{F,t+1})(C^H_t + P^H_t) + (1 + \theta)(1 + r_{S,t+1})S^H_t - (1 - \phi)\Delta L^H_{t+1}\Big] \\
&- \Big[(1 + r_{I,t+1})(D^H_t - D^P_t) - (1 + r_{F,t+1})(C^H_t + P^P_t) + (1 + \theta')(1 + r_{S,t+1})S^H_t \\
&- (1 - \phi)\Delta L^H_{t+1} + (1 + \phi')[(1 + r_{C,t+1})D^P_t - D^P_{t+1}]\Big] - D^P_{t+1} + (1 + r_{I,t+1})(P^H_t - P^P_t) \\
&+ [(1 + r_{I,t+1})C^P_t + (1 + r_{M,t+1})A^P_t],
\end{aligned}$$

so that the return on equity is further reduced by $\phi'[(1 + r_{C,t+1})D^P_t - D^P_{t+1}]/N^P_t$:

$$\begin{aligned}
1 + r_{NP,t+1} &= \frac{N^H_t}{N^P_t}\Bigg\{\Bigg[- (1 + r_{C,t+1})\rho_t\alpha_t\gamma_t\frac{1 - \mu^P_{L,t}}{\mu_{L,t}} - (1 + r_{I,t+1})\alpha_t\gamma_t\frac{1 - \mu_{L,t} - \rho_t(1 - \mu^P_{L,t})}{\mu_{L,t}} \\
&+ (r_{I,t+1} - r_{F,t+1})\alpha_t(1 - \gamma_t)\frac{\mu_{S,t} - \mu^P_{S,t}}{\mu_{S,t}}\Bigg] \\
&+ \frac{\theta - \theta'}{m_{S,t} - \theta}\Bigg[(1 + r_{D,t+1})\frac{m_{S,t} - \theta'}{\theta - \theta'}\frac{\alpha_t\gamma_t(1 - \mu_{L,t})}{\mu_{L,t}} - (1 + r_{L,t+1})(1 - m_{L,t})\frac{\alpha_t\gamma_t}{\mu_{L,t}} \\
&+ (1 + r_{S,t+1})(1 + m_{S,t})\frac{\alpha_t(1 - \gamma_t)}{\mu_{S,t}} - (1 + r_{F,t+1})\frac{\alpha_t(1 - \gamma_t) + \mu_{S,t}(1 - \alpha_t\gamma_t)}{\mu_{S,t}}\Bigg] \\
&- \frac{\phi'\rho_t(1 - m^P_{L,t})}{m_{L,t} - \phi}\Bigg[(1 + r_{C,t+1})\frac{m_{L,t} - \phi}{1 - m^P_{L,t}}\alpha_t\gamma_t\frac{1 - \mu^P_{L,t}}{\mu_{L,t}} - (1 + r_{L,t+1})(1 - \phi)\frac{\alpha_t\gamma_t}{\mu_{L,t}} \\
&+ (1 + r_{S,t+1})(1 + \theta)\frac{\alpha_t(1 - \gamma_t)}{\mu_{S,t}} + (1 + r_{D,t+1})\frac{\alpha_t\gamma_t(1 - \mu_{L,t})}{\mu_{L,t}} \\
&\quad -(1 + r_{F,t+1})\frac{\alpha_t(1 - \gamma_t) + \mu_{S,t}(1 - \alpha_t\gamma_t)}{\mu_{S,t}}\Bigg]\Bigg\} \\
&+ [(1 + r_{I,t+1})(1 - \alpha^P_t) + (1 + r_{M,t+1})\alpha^P_t].
\end{aligned}$$

After some straightforward simplification, we obtain that:

$$\begin{aligned}
1+r_{NP,t+1} &= \frac{N_t^H}{N_t^P}\Bigg[-(1+\phi')(1+r_{C,t+1})\rho_t\alpha_t\gamma_t\frac{1-\mu_{L,t}^P}{\mu_{L,t}} \\
&- (1+r_{I,t+1})\alpha_t\gamma_t\frac{1-\mu_{L,t}-\rho_t(1-\mu_{L,t}^P)}{\mu_{L,t}} + (r_{I,t+1}-r_{F,t+1})\alpha_t(1-\gamma_t)\frac{\mu_{S,t}-\mu_{S,t}^P}{\mu_{S,t}}\Bigg] \\
&+ (1+r_{D,t+1})\frac{\alpha_t\gamma_t(1-\mu_{L,t})}{\mu_{L,t}}\left(\frac{m_{S,t}-\theta'}{m_{S,t}-\theta}-\frac{\phi'(1-m_{L,t}^P)\rho_t}{m_{L,t}-\phi}\right) \\
&- (1+r_{L,t+1})\frac{\alpha_t\gamma_t}{\mu_{L,t}}\left(\frac{\theta-\theta'}{m_{S,t}-\theta}(1-m_{L,t})-\frac{\phi'(1-m_{L,t}^P)}{m_{L,t}-\phi}(1-\phi)\rho_t\right) \\
&+ (1+r_{S,t+1})\frac{\alpha_t(1-\gamma_t)}{\mu_{S,t}}\left(\frac{\theta-\theta'}{m_{S,t}-\theta}(1+m_{S,t})-\frac{\phi'(1-m_{L,t}^P)}{m_{L,t}-\phi}(1+\theta)\rho_t\right) \\
&- (1+r_{F,t+1})\frac{\alpha_t(1-\gamma_t)+\mu_{S,t}(1-\alpha_t\gamma_t)}{\mu_{S,t}}\left(\frac{\theta-\theta'}{m_{S,t}-\theta}-\frac{\phi'(1-m_{L,t}^P)\rho_t}{m_{L,t}-\phi}\right)\Bigg] \\
&+ [(1+r_{I,t+1})(1-\alpha_t^P)+(1+r_{M,t+1})\alpha_t^P].
\end{aligned}$$

We notice that, as switching from secured to unsecured financing is costly ($\phi' > 0$), the return on equity of the prime broker is reduced when the rehypothecation rate is large and the hedge fund has to delever.

4.2.5 Default of the hedge fund

When the market return is below $\bar{r}_{M,t+1}^{(LS)}$, the hedge fund cannot fully repay its debt and therefore defaults. In this extreme case, the amount of debt that the hedge fund can repay is given by:

$$D_{t+1}^{H(DE)} = (1+r_{F,t+1})(C_t^H+P_t^H)+(1-\phi)(1+r_{L,t+1})L_t^H-(1+\theta)(1+r_{S,t+1})S_t^H.$$

The deleveraging of the hedge fund has two effects on the debt of the prime broker. On the one hand, the unsecured debt is reduced by the same amount $(1+r_{F,t+1})(C_t^H+P_t^H)+(1-\phi)(1+r_{L,t+1})L_t^H-(1+\theta)(1+r_{S,t+1})S_t^H$. On the other hand, as the hedge fund sells all its securities, there is no collateral left to the prime broker for its secured debt. The prime broker has to switch from secured to unsecured debt for an amount equal to its initial secured debt plus the cost of switching, i.e., $(1+\phi')(1+r_{C,t+1})D_t^P$.

$$
\begin{aligned}
N^P_{t+1} &= \Big[(1+r_{D,t+1})D^H_t - (1+r_{F,t+1})(C^H_t + P^H_t) + (1+\theta)(1+r_{S,t+1})S^H_t - (1-\phi)\Delta L^H_{t+1}\Big] \\
&- \Big[(1+r_{I,t+1})(D^H_t - D^P_t) - (1+r_{F,t+1})(C^H_t + P^P_t) + (1+\theta')(1+r_{S,t+1})S^H_t \\
&- (1-\phi)\Delta L^H_{t+1} + (1+\phi')[(1+r_{C,t+1})D^P_t - D^P_{t+1}]\Big] - D^P_{t+1} + (1+r_{I,t+1})(P^H_t - P^P_t) \\
&+ [(1+r_{I,t+1})C^P_t + (1+r_{M,t+1})A^P_t],
\end{aligned}
$$

In this situation, all the positions of the hedge fund are liquidated, resulting in the following expression for the equity of the prime broker at $t+1$:

$$
\begin{aligned}
N^P_{t+1} &= \Big[(1+r_{D,t+1})D^H_t - (1+r_{F,t+1})(C^H_t + P^H_t) + (1+\theta)(1+r_{S,t+1})S^H_t - (1-\phi)(1+r_{L,t+1})L^H_t\Big] \\
&- \Big[(1+r_{I,t+1})(D^H_t - D^P_t) - (1+r_{F,t+1})(C^H_t + P^P_t) + (1+\theta')(1+r_{S,t+1})S^H_t \\
&- (1-\phi)(1+r_{L,t+1})L^H_t + (1+\phi')(1+r_{C,t+1})D^P_t\Big] \\
&+ (1+r_{I,t+1})(P^H_t - P^P_t) + [(1+r_{I,t+1})C^P_t + (1+r_{M,t+1})A^P_t], \\
&= \Big[(1+r_{D,t+1})D^H_t - (1+r_{C,t+1})D^P_t - (1+r_{I,t+1})(D^H_t - D^P_t) \\
&+ (r_{I,t+1} - r_{F,t+1})(P^H_t - P^P_t) + (1+r_{I,t+1})C^P_t + (1+r_{M,t+1})A^P_t\Big] \\
&- \phi'(1+r_{C,t+1})D^P_t + (\theta - \theta')(1+r_{S,t+1})S^H_t.
\end{aligned}
$$

As the terms in squared brackets correspond to the equity of the prime broker in normal time, the last two terms measure the loss incurred by the prime broker due to the hedge fund's default.

We obtain the return on equity:

$$
\begin{aligned}
1 + r_{NP,t+1} &= \frac{N^H_t}{N^P_t}\Big[(1+r_{D,t+1})\alpha_t\gamma_t\frac{1-\mu_{L,t}}{\mu_{L,t}} - (1+\phi')(1+r_{C,t+1})\rho_t\alpha_t\gamma_t\frac{1-\mu^P_{L,t}}{\mu_{L,t}} \\
&- (1+r_{I,t+1})\alpha_t\gamma_t\frac{1-\mu_{L,t}-\rho_t(1-\mu^P_{L,t})}{\mu_{L,t}} + (r_{I,t+1}-r_{F,t+1})\alpha_t(1-\gamma_t)\frac{\mu_{S,t}-\mu^P_{S,t}}{\mu_{S,t}} \\
&+ (\theta-\theta')(1+r_{S,t+1})\frac{\alpha_t(1-\gamma_t)}{\mu_{S,t}}\Big] + [(1+r_{I,t+1})(1-\alpha^P_t) + (1+r_{M,t+1})\alpha^P_t].
\end{aligned}
$$

The prime broker defaults if its equity becomes negative, i.e., $r_{N_P,t+1} < -1$:

$$
\begin{aligned}
&(1+r_{D,t+1})\alpha_t\gamma_t\frac{1-\mu_{L,t}}{\mu_{L,t}} - (1+\phi')(1+r_{C,t+1})\rho_t\alpha_t\gamma_t\frac{1-\mu^P_{L,t}}{\mu_{L,t}} \\
&-(1+r_{I,t+1})\alpha_t\gamma_t\frac{1-\mu_{L,t}-\rho_t(1-\mu^P_{L,t})}{\mu_{L,t}} + (r_{I,t+1}-r_{F,t+1})\alpha_t(1-\gamma_t)\frac{\mu_{S,t}-\mu^P_{S,t}}{\mu_{S,t}} \\
&+(\theta-\theta')(1+r_{S,t+1})\frac{\alpha_t(1-\gamma_t)}{\mu_{S,t}} < -\frac{N^P_t}{N^H_t}\left[1+(1+r_{I,t+1})(1-\alpha^P_t)+(1+r_{M,t+1})\alpha^P_t\right].
\end{aligned}
$$

This condition can be written in terms of the market return:

$$
\begin{aligned}
&r_{M,t+1}\left((\theta-\theta')\frac{\alpha_t(1-\gamma_t)}{\mu_{S,t}}\beta_S + \frac{N^P_t}{N^H_t}\alpha^P_t\right) < -\frac{N^P_t}{N^H_t}\left[1+(1+r_{I,t+1})(1-\alpha^P_t)\right] \\
&-(1+r_{D,t+1})\alpha_t\gamma_t\frac{1-\mu_{L,t}}{\mu_{L,t}} + (1+r_{C,t+1})(1+\phi')\rho_t\alpha_t\gamma_t\frac{1-\mu^P_{L,t}}{\mu_{L,t}} \\
&+(1+r_{I,t+1})\alpha_t\gamma_t\frac{1-\mu_{L,t}-\rho_t(1-\mu^P_{L,t})}{\mu_{L,t}} - (r_{I,t+1}-r_{F,t+1})\alpha_t(1-\gamma_t)\frac{\mu_{S,t}-\mu^P_{S,t}}{\mu_{S,t}} \\
&-(\theta-\theta')\frac{\alpha_t(1-\gamma_t)}{\mu_{S,t}} - \frac{N^P_t}{N^H_t}\alpha^P_t.
\end{aligned}
$$

When $\theta = \theta'$, the default threshold for the prime broker depends on the performance of the market only through the securities portfolio. When $\theta = \theta'$, $\mu^P_{S,t} = \mu_{S,t}$, $\mu^P_{L,t} = \mu_{L,t}$, and $\rho = 1$, we obtain:

$$
\begin{aligned}
1+r_{NP,t+1} &= \frac{N^H_t}{N^P_t}\frac{\alpha_t\gamma_t(1-\mu_{L,t})}{\mu_{L,t}}[(1+r_{D,t+1})-(1+r_{C,t+1})(1+\phi')] \\
&+ [(1+r_{I,t+1})(1-\alpha^P_t)+(1+r_{M,t+1})\alpha^P_t].
\end{aligned}
$$

When $\theta = \theta'$, $\mu^P_{S,t} = \mu_{S,t}$, $\mu^P_{L,t} = \mu_{L,t}$, and $\rho = 0$, we obtain:

$$
\begin{aligned}
1+r_{NP,t+1} &= \frac{N^H_t}{N^P_t}\frac{\alpha_t\gamma_t(1-\mu_{L,t})}{\mu_{L,t}}[r_{D,t+1}-r_{I,t+1}] \\
&+ [(1+r_{I,t+1})(1-\alpha^P_t)+(1+r_{M,t+1})\alpha^P_t].
\end{aligned}
$$

Figure 10 illustrates the value of the return on equity of the hedge fund and the prime broker as a function of the market return with the parametrization reported in Table 1. The prime broker is affected by the market downturn through two channels: (1) the loss incurred by the hedge fund in its deleveraging process, which spills over the prime broker;

(2) the market portfolio held by the prime broker also loses value. As the figure shows, the slope of the prime broker's curve is steeper when the hedge fund liquidates its long positions (for market return between −14.8% and −27%) because it also forces the prime broker to reduces its own secured debt based on rehypothecated collateral.

4.3 Optimal Strategy

The interbank margin rates ($\mu^P_{L,t}$, $\mu^P_{S,t}$, and $m^P_{L,t}$), the risk-free rate ($r_{F,t+1}$), the general collateral rate ($r_{C,t+1}$), the interbank short-term borrowing rate ($r_{I,t+1}$), and the distribution of the market return (μ_M and σ_M) are given to the prime broker. Then, the prime broker determines its optimal strategy, i.e., the fraction of its equity held in cash ($1-\alpha^P_t$), the rehypothecation rate (ρ_t), and the initial and maintenance margin rates applied to the hedge fund ($\mu_{L,t}$, $\mu_{S,t}$, $m_{L,t}$, and $m_{S,t}$). The optimal parameters are found by maximizing the expression:

$$\max_{\{\alpha^P_t,\rho_t,\mu_{L,t},\mu_{S,t},m_{L,t},m_{S,t}\}} E_t[r_{NP,t+1}],$$

with the restrictions $0 \leq \alpha^P_t, \rho_t, \mu_{L,t}, \mu_{S,t}, m_{L,t}, m_{S,t} \leq 1$.

The lending rate $r_{D,t+1}$ is determined in the next section from the risk premium over the risk-free rate paid by the hedge fund to borrow funds from the prime broker.

5 Equilibrium Lending Rate

Thus far, we have assumed that the lending rate, $r_{D,t+1}$, for a loan of the prime broker to the hedge fund is given at the beginning of period t. However, because the hedge fund can default, the loan is risky for the prime broker. In the case of a hedge fund's default, the prime broker has to liquidate the hedge fund's assets. The prime broker should take this potential loss into account and require a risk premium. Assuming that the prime broker is risk neutral, the risk premium must cover the expected loss on the loan due to the possible hedge fund's default.

The expected value of the loan for the prime broker is given by:

$$E_t[D^H_{t+1}] = (1 + r_{D,t+1})D^H_t \times \Pr[r_{M,t+1} > \bar{r}^{(DE)}_{M,t+1}] + E_t[D^{(DE)}_{t+1}] \times \Pr[r_{M,t+1} \leq \bar{r}^{(DE)}_{M,t+1}],$$

where $E_t[D_{t+1}^{(DE)}]$ corresponds to the expected value of the debt that the hedge fund is not able to repay in case of default, i.e.:

$$E_t[D_{t+1}^{(DE)}] = (1+r_{F,t+1})(C_t^H + P_t^H) + (1+\beta_L \mu_{M,t+1}^{(DE)})(1-\phi)L_t^H - (1+\theta)(1+\beta_S \mu_{M,t+1}^{(DE)})S_t^H,$$

where

$$\mu_{M,t+1}^{(DE)} = E_t[r_{M,t+1} \mid r_{M,t+1} \leq \bar{r}_{M,t+1}^{(DE)}] = \exp\left(\mu_M + (1/2)\sigma_M^2\right) \frac{F(\log(1+\bar{r}_{M,t+1}^{(DE)}))}{G(\log(1+\bar{r}_{M,t+1}^{(DE)}))}$$

denotes the expected market return corresponding to a hedge fund's default. Therefore, the expected value of the loan is as follows:

$$\begin{aligned} E_t[D_{t+1}^H] &= (1+r_{D,t+1})D_t^H - \Big[(1+r_{D,t+1})D_t^H - (1+r_{F,t+1})(C_t^H + P_t^H) \\ &- (1+\beta_L \mu_{M,t+1}^{(DE)})(1-\phi)L_t^H + (1+\theta)(1+\beta_S \mu_{M,t+1}^{(DE)})S_t^H\Big] F(\log(1+\bar{r}_{M,t+1}^{(DE)})). \end{aligned}$$

In the case of a hedge fund's default, the value of the loan is equal to the face value of the debt reduced by the loss that affects the hedge fund's assets.

Given the possibility of an ex-post loss on the loan, the prime broker requires ex-ante an interest rate $r_{D,t+1}$ that takes the expected loss into account. Therefore, we have $r_{D,t+1} = r_{F,t+1} + RP_{D,t+1}$, where $RP_{D,t+1}$ corresponds to the risk premium over the risk-free rate $r_{F,t+1}$. The risk premium is defined such that it covers the expected loss in case of default:

$$\begin{aligned} RP_{D,t+1} &= -E_t\left[\frac{D_{t+1}^H - (1+r_{D,t+1})D_t^H}{D_t^H}\right] \\ &= \Big[(1+r_{D,t+1}) - (1+r_{F,t+1})\frac{[\alpha_t(1-\gamma_t) + \mu_{S,t}(1-\alpha_t\gamma_t)]\mu_{L,t}}{\alpha_t\gamma_t(1-\mu_{L,t})\mu_{S,t}} \\ &- (1+\beta_L \mu_{M,t+1}^{(DE)})\frac{1-\phi}{1-\mu_{L,t}} + (1+\beta_S \mu_{M,t+1}^{(DE)})\frac{(1+\theta)\alpha_t(1-\gamma_t)\mu_{L,t}}{\alpha_t\gamma_t(1-\mu_{L,t})\mu_{S,t}}\Big] F(\log(1+\bar{r}_{M,t+1}^{(DE)})). \end{aligned}$$

The risk premium increases with leverage through two channels: both the probability of the hedge fund's default and the loss on the loan in the event of a default increase with leverage.

At equilibrium, the hedge fund's optimal decision about the amount of collateral (α_t^* and γ_t^*) and the prime broker's optimal decision about the amount of margin rates ($\mu_{L,t}^*$ and $\mu_{S,t}^*$) must be consistent with the equilibrium lending rate, i.e., $r_{D,t+1}$ should satisfy the following equation:

$$\begin{aligned} r_{D,t+1} &= r_{F,t+1} + \Bigg[(1+r_{D,t+1}) - (1+r_{F,t+1})\frac{[\alpha_t^*(1-\gamma_t^*)+\mu_{S,t}^*(1-\alpha_t^*\gamma_t^*)]\mu_{L,t}^*}{\alpha_t^*\gamma_t^*(1-\mu_{L,t}^*)\mu_{S,t}^*} \\ &- (1+\beta_L\mu_{M,t+1}^{(DE)})\frac{1-\phi}{1-\mu_{L,t}^*} + (1+\beta_S\mu_{M,t+1}^{(DE)})\frac{(1+\theta)\alpha_t^*(1-\gamma_t^*)\mu_{L,t}^*}{\alpha_t^*\gamma_t^*(1-\mu_{L,t})\mu_{S,t}^*}\Bigg] F(\log(1+\bar{r}_{M,t+1}^{(DE)})), \end{aligned} \tag{5}$$

where the threshold $\bar{r}_{M,t+1}^{(DE)}$ also depends on the equilibrium lending rate and the decisions variables of the hedge fund and the prime broker. Therefore, the equilibrium lending rate, $r_{D,t+1}$, is determined as a fixed point of Equation (5). We note that $\mu_{M,t+1}^{(DE)}$, $\bar{r}_{M,t+1}^{(DE)}$, α_t^*, γ_t^*, $\mu_{L,t}^*$, and $\mu_{S,t}^*$ depend on $r_{D,t+1}$ at the equilibrium. Therefore, the optimal leverage of the hedge fund (through α_t and γ_t), the optimal leverage of the prime broker (through α_t^P and ρ_t), the margin rates ($\mu_{L,t}$, $\mu_{S,t}$, $m_{L,t}$, and $m_{S,t}$) and the equilibrium lending rate ($r_{D,t+1}$) are determined simultaneously for a given set of interbank margin rates ($\mu_{S,t}^P$, $\mu_{L,t}^P$, and $m_{L,t}^P$) and the exogenous interest rates ($r_{F,t+1}$, $r_{I,t+1}$, $r_{C,t+1}$).

6 Empirical Analysis

6.1 Strategy to Compute the Optimal Decisions

As described in Sections 3 to 5, the model ultimately determines the optimal financing decisions of the hedge fund and the prime broker and the equilibrium lending rate. Exogenous rates are: the risk-free rate ($r_{F,t+1}$), the general collateral rate ($r_{C,t+1}$), the short-term unsecured borrowing rate $r_{I,t+1}$, and the distribution of the market return (μ_M and σ_M). Exogenous parameters are: the investment strategy of the hedge fund (β_L and β_S), the liquidation cost for short positions (θ and θ') and long positions (ϕ and ϕ'), and the margin rates for the prime broker ($\mu_{L,t}^P$, $\mu_{S,t}^P$, and $m_{L,t}^P$). We denote by $\xi_H^* = (\alpha^*, \gamma^*)'$ and $\xi_P^* = (\alpha^{P*}, \rho^*, \mu_L^*, \mu_S^*, m_L^*, m_S^*)'$ the optimal decisions variables of the hedge fund and the prime broker.

The optimal decision variables of the hedge fund are first determined for given values of the prime broker decision variables and the lending rate, i.e., $\xi_H^*(\xi_P^*, r_D)$. Then, the optimal decisions of the prime broker are determined for given values of the lending rate, i.e., $\xi_P^*(r_D)$. Finally, the equilibrium lending rate (r_D^*) is obtained by solving Equation (5). Therefore, the optimal solution (ξ_H^*, ξ_P^*, and r_D^*) is obtained by successive iterations of the three optimization programs: The main optimization is on the equilibrium lending rate. For a given guess on r_D, we run the second optimization of the prime broker's program. For a given guess on ξ_P, we run the third optimization of the hedge fund's program. Once we obtain the optimal value for ξ_H^*, we can compute the value of ξ_P^* and consequently of r_D^*. Then we iterate on r_D to solve Equation (5).

As mentioned above, some parameters are calibrated because they do not depend on the decision of the hedge fund and the prime broker (Table 1). The risk-free rate ($r_{F,t+1}$), the general collateral rate ($r_{C,t+1}$), and the short-term unsecured borrowing rate ($r_{I,t+1}$) are set equal to 1.5%, 1.55%, and 1.75% per year.The parameters of the distribution of the market return are equal to $\mu_M = 2\%$ and $\sigma_M = 22.5\%$ per year. The various costs of liquidation are equal to $\phi = 5\%$ and $\theta = 2\%$ for the hedge fund and to $\phi' = 2.5\%$ and $\theta' = 1\%$ for the prime broker.The margin rates for the prime broker are given by $\mu_L^P = \mu_S^P = 20\%$ and $m_L^P = m_S^P = 10\%$.The beta parameters corresponding to the long and short strategies of the hedge fund are given by $\beta_L = 1.1$ and $\beta_S = 0.2$, meaning that the long positions will suffer more from the market crash than the short positions will benefit from it. Finally, the baseline value for the leverage limit for the prime broker is set to $\vartheta = 10$.

6.2 Results

6.2.1 Benchmark Case

We start with the optimal decisions of the hedge fund and the prime broker in the benchmark case, with parameters calibrated as in Table 1. In solving the optimization process described in the previous section, we find the optimal values (ξ_H^*, ξ_P^*, and r_D^*).

The equilibrium lending rate is equal to 1.552% per year, which is slightly above the general collateral rate. We notice that the risk premium over the general collateral rate is rather low. The reason for this result is that the default threshold is equal to $\bar{r}_{t+1}^{(DE)} = -25.6\%$, which corresponds to a probability of default of the hedge fund equal to 0.4%.

The optimal decision of the prime broker is $\alpha^{P*} = 0.736$ and $\gamma^* = 0.698$. These values mean that the prime broker holds a significant amount of free cash and rehypothecates approximately 70% of the collateral deposited by the hedge fund. The initial margin rate for imposed to the hedge fund is set equal to 26.4%, whereas the maintenance margin rate is set equal to 18%. We notice that they are well below the values fixed by the Regulation T, but above the values assumed for the margin rates imposed to the prime broker collateralized transactions.

The margin multiplier of the hedge fund is equal to $\alpha^* = 0.923$ and the fraction of long positions is equal to $\gamma^* = 0.902$, i.e., 90% of the equity is invested on long positions. These values, close to those we found in the preliminary analysis in Section 3, suggest that the hedge fund relies on free cash to reduce its risk of default (and therefore its lending rate) but also combines long and short positions to reduce its exposure to the market risk. Schemata 5 and 6 displays the balance sheet of the hedge fund and the prime broker.The hedge fund's equity has been arbitrarily fixed at $15,000$.

Given the optimal decisions of the hedge fund and the prime broker, we also determine the expected return on equity of both agents. The hedge fund is expected to generate a return on equity of 11.4% per year, whereas the expected return on equity of the prime broker is equal 3.4% per year.

Schema 5: Balance sheet of a hedge fund at date t

	Assets		Liabilities and Equity		
(1.5%)	Free cash	1153	Margin debt	34898	($r_{D,t+1}$)
(1.5%)	Cash proceeds	6498	Short securities	5142	($r_{S,t+1}$)
($r_{L,t+1}$)	Long securities	47390	Equity	15000	($r_{NH,t+1}$)
			- Free cash	1153	
			- Long account	12492	
			- Short account	1356	
	Total	55040	**Total**	55040	

Schema 6: Balance Sheet of a Prime Broker

	Assets		Liabilities and Equity		
(1.75%)	Unsecured Cash	1246	Unsecured borrowing	1806	(1.75%)
(1.5%)	Sec. borrowed	6171	Sec. loaned	6498	(1.5%)
($r_{D,t+1}$)	Receivables from HF	34898	Payables to investor	26473	(1.55%)
($r_{M,t+1}$)	Sec. portfolio	2559	Equity	3478	($r_{NP,t+1}$)
			- Free cash	3478	
			- Other	919	
	Total	44873	**Total**	44873	

6.2.2 Impact of decreasing prime broker's leverage

Decreasing the level of leverage that the prime broker is allowed to built (from $\vartheta = 10$ to 5) has several implications in our model (see Table 2). First, to maximize its expected return on equity, the prime broker decreases the margin rates to increase its revenues from the hedge fund. The hedge fund in turn increases its leverage. To increase its own return on equity, the hedge fund also reduces the amount of free cash. As a consequence, its probability of default increases (from 0.4% to 0.9%) and so does the lending rate (from 1.552% per year to 1.638%). Given the higher risk induces by the hedge fund financing, the prime broker also increases the amount of free cash and reduces the amount of secured borrowing based on rehypothecated collateral. In the end, due to the higher leverage of the hedge fund, the return on equity is increased for both the hedge fund (from 11.46% to 11.83%) and the prime broker (from 3.35% to 3.59%).

6.2.3 Impact of increasing the market volatility

As Table 2 also shows, a rise in the market volatility (from 22.5% to 23% per year) increases the overall risk in the model. Therefore, the prime broker increases the margin rates as an attempt to mitigate the higher risk. In doing so, the hedge fund is incentivized to take on more risk, by reducing its free cash and short positions to increase its return on equity. As the expected loss of the prime broker in case of default of the hedge fund is higher, the prime broker increases the amount of free cash it holds. The return on equity is reduced for both agents.

7 Conclusion

In our model, the hedge fund optimally chooses the amount of free cash and the combination of long and short strategies. The prime broker optimally chooses amount of free cash, the margin rates, and the rehypothecation rate. The implication of the analysis for hedge fund managers is that when they choose a prime broker or deal with existent ones they should carefully consider the incentives of the prime broker. The choice of a prime broker is getting even more crucial in light of the regulation which will be imposed by the Basel III. Prime brokers will have to put aside a lot more capital when they lend to hedge funds. Hedge fund managers should understand the value that their business represents. The metrics prime brokers use to evaluate hedge fund portfolios is not publicly disclosed by them, but it is of interest to regulators that control financial market stability.

A promising direction for future work is to introduce prime brokerage commission. Also, the paper raises the question how Basel III affects hedge fund and prime brokers, i.e. the impact on the dynamics between lenders and borrowers.

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A Appendix

A.1 Functioning of Prime Broker

To illustrate how the balance sheet is affected by the decision of the prime broker to finance hedge fund investment, we consider the case of a long position (Schema 4) and a short position (Schema 5).

A long position by a hedge fund financed on margin works as follows: (1) The prime broker lends 90 of cash to the hedge fund, with a margin rate of $\mu_L = 10\%$. (2) With the 90 of the loan and 10 of its own equity, the hedge fund buys 100 of securities. (3) The prime broker can re-use the securities for rehypothecation by lending the 100 of securities to a borrower, with a margin rate of $\mu_P = 5\%$. In such a case, the prime broker receives 100 of cash and a deposit of 5 from the borrower.

Schema 4: Financing a long position of the hedge fund

Assets		**Liabilities**	
Cash	$-90 + 100$	Collateralized Financings:	
Cash and Securities Segregated:		- Securities loaned to Borrower	100
- from Hedge fund (Long account)	10	Payables:	
- from Borrowers (Short account)	5	- to Hedge fund (Deposit)	10
Receivables:		- to Borrowers (Deposit)	5
- from Hedge fund (Loan)	90	Equity	

A short position by a hedge fund financed on margin works as follows: (1) The prime brokers borrows 100 of securities from a lender (or enters a reverse repo) with margin rate of $\mu_P = 5\%$. It posts 5 of its equity in deposit on the lender margin account. (2) The prime broker lends the 100 of securities to the hedge fund. The hedge fund deposits 10 of cash of its own equity on the prime broker margin account, with $\mu_S = 10\%$. (3) The hedge fund sells the securities and let the cash proceeds of 100 on the prime broker account.

Schema 5: Financing a short position of the hedge fund

Assets		Liabilities	
Cash and Securities Segregated:		Payables:	
- from Hedge fund (Short account)	10	- to Hedge fund (Deposit)	10
Collateralized agreements:		- to Hedge fund (Cash proceeds)	100
- Securities borrowed from Lender	100	Equity	
Receivables:		- Short account	5
- from Lender (Deposit)	5		

Table 1: Value of the calibrated parameters

	Symbol	Value
Hedge fund		
Liquidation cost for long positions	ϕ	5%
Liquidation cost for short positions	θ	2%
Initial margin rate	$\mu_L = \mu_S$	30%
Maintenance margin rate	$m_L = m_S$	20%
Sensitivity of long positions	β_L	1.1
Sensitivity of short positions	β_S	0.2
Prime broker		
Liquidation cost for long positions	ϕ'	2.5%
Liquidation cost for short positions	θ'	1%
Initial margin rate	$\mu_L^P = \mu_S^P$	20%
Maintenance margin rate	$m_L^P = m_S^P$	10%
Maximum leverage ratio	ϑ	10
Market return and interest rates		
Expected market return	μ_M	2%
Market volatility	σ_M	22.5%
Risk-free rate	r_F	1.5%
General collateral rate	r_C	1.55%
Short-term interbank rate	r_I	1.75%

Note: The table reports the baseline value of the model's parameters.

Table 2: Optimal decisions parameters for alternative parameter scenarios

	Benchmark	$\vartheta = 5$	$\sigma_M = 23\%$
Hedge fund's decision variables			
α^*	0.923	0.945	0.936
γ^*	0.902	0.924	0.958
Prime broker's decision variables			
$\mu_L^* = \mu_S^*$	0.264	0.254	0.292
$m_L^* = m_S^*$	0.180	0.154	0.197
α^{P*}	0.736	0.677	0.695
ρ^*	0.698	0.556	0.709
Rates and expected returns (% per year)			
$r_{D,t+1}^*$	1.550	1.648	1.550
$E_t[r_{NH,t+1}]$	11.458	11.832	11.414
$E_t[r_{NP,t+1}]$	3.351	3.590	3.167
Probability of default (in %)			
$\Pr[r_{M,t+1} < \bar{r}_{M,t+1}^{(DE)}]$	0.373	0.893	0.348

Figure 1: Value of the hedge fund's return on equity as a function of the market return

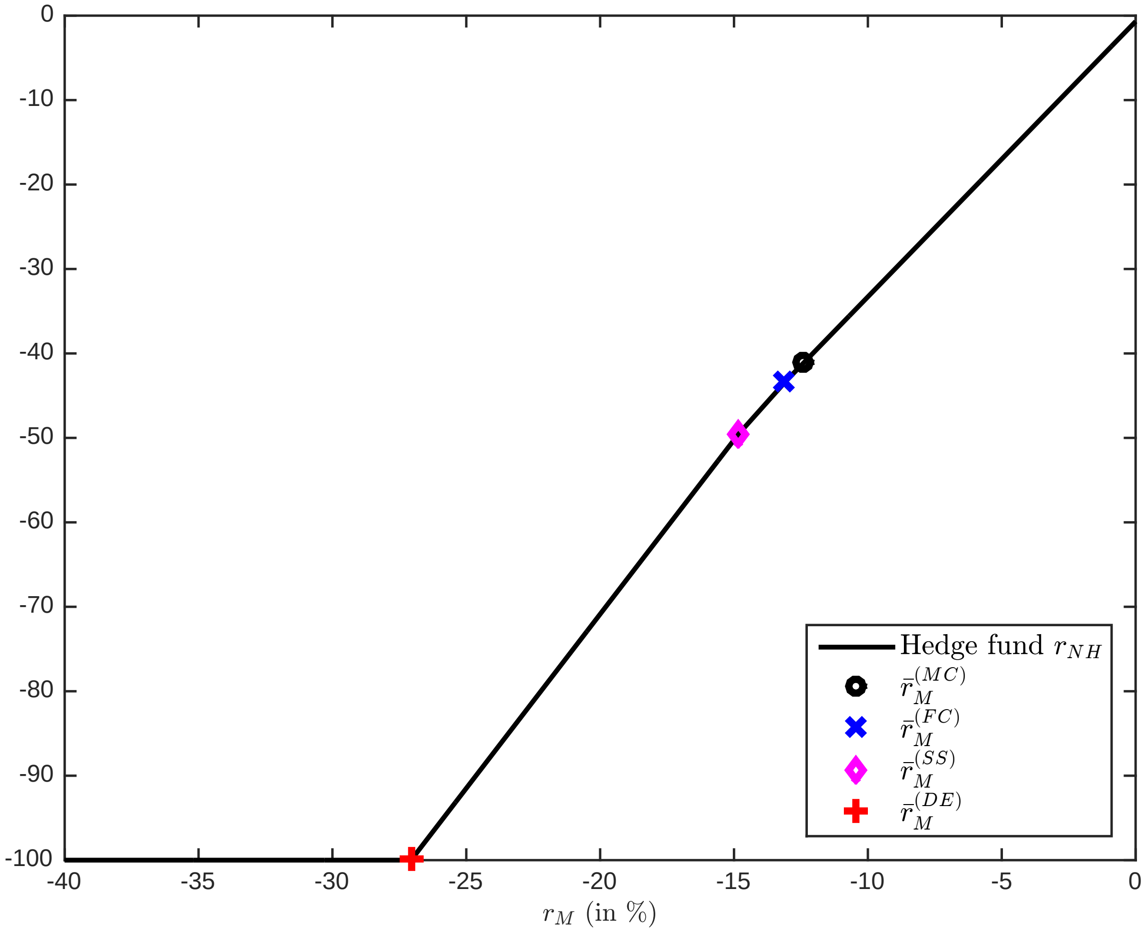

Figure 2: Hedge fund's decisions as a function of expected market return

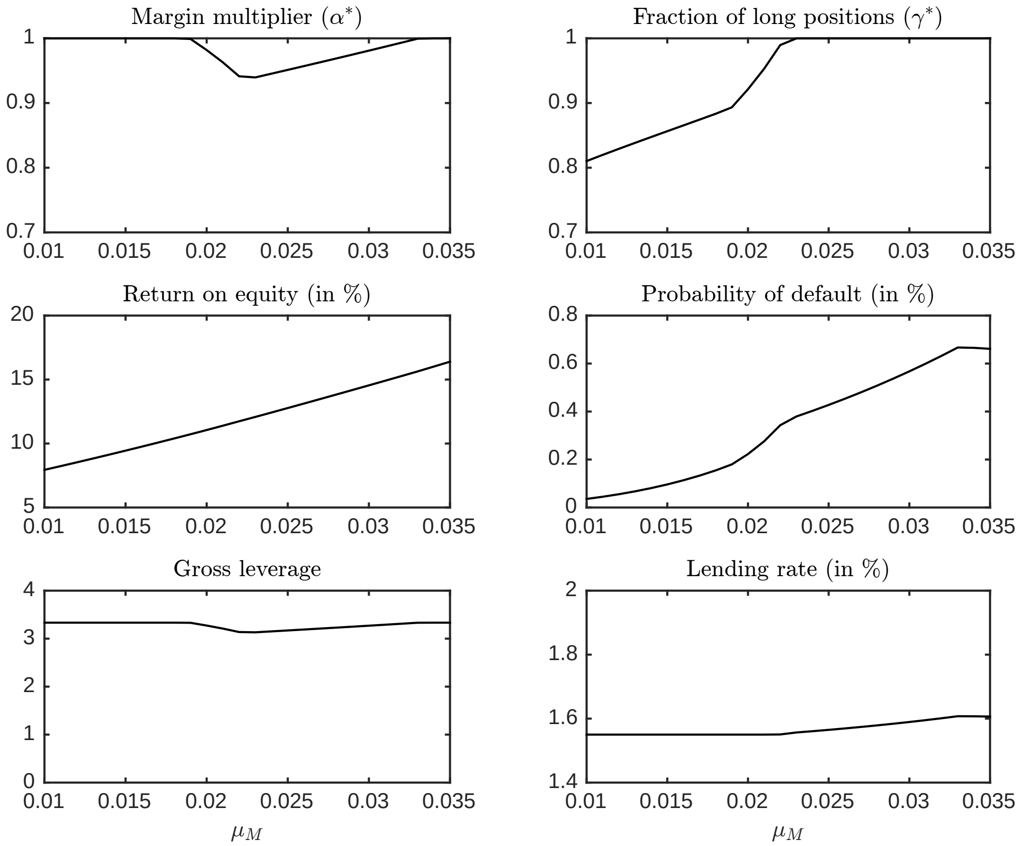

Figure 3: Hedge fund's decisions as a function of market volatility

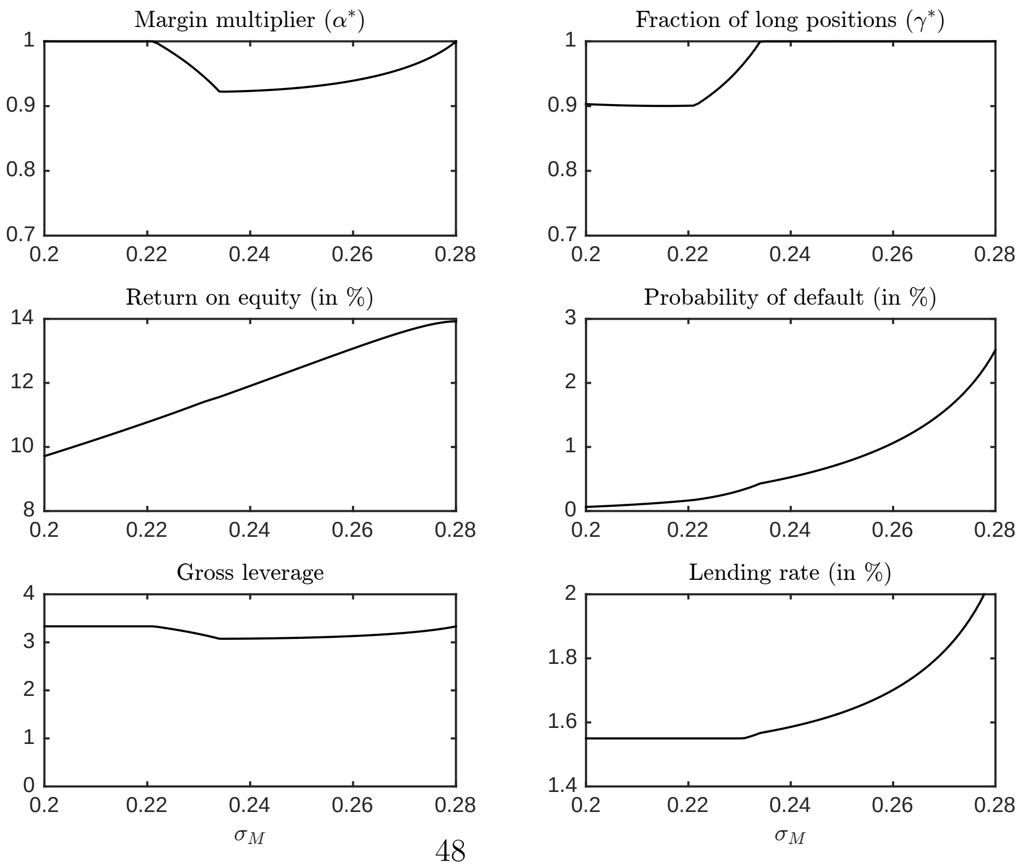

Figure 4: Hedge fund's decisions as a function of β_L

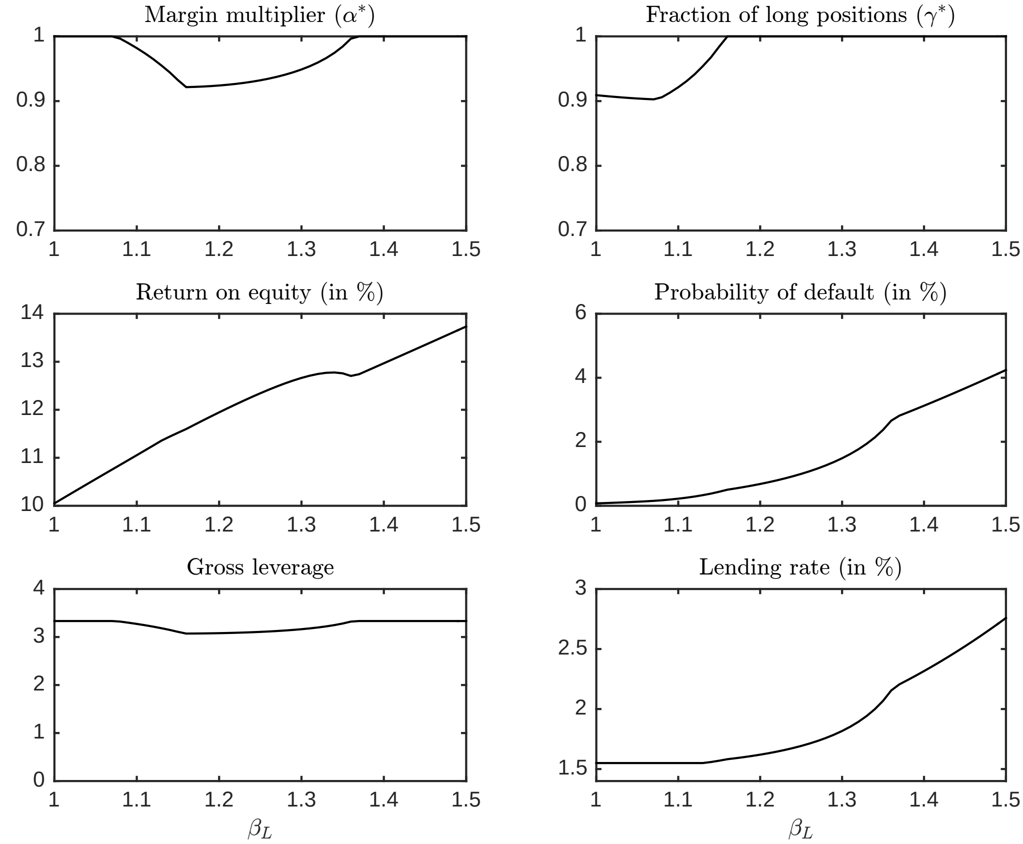

Figure 5: Hedge fund's decisions as a function of β_S

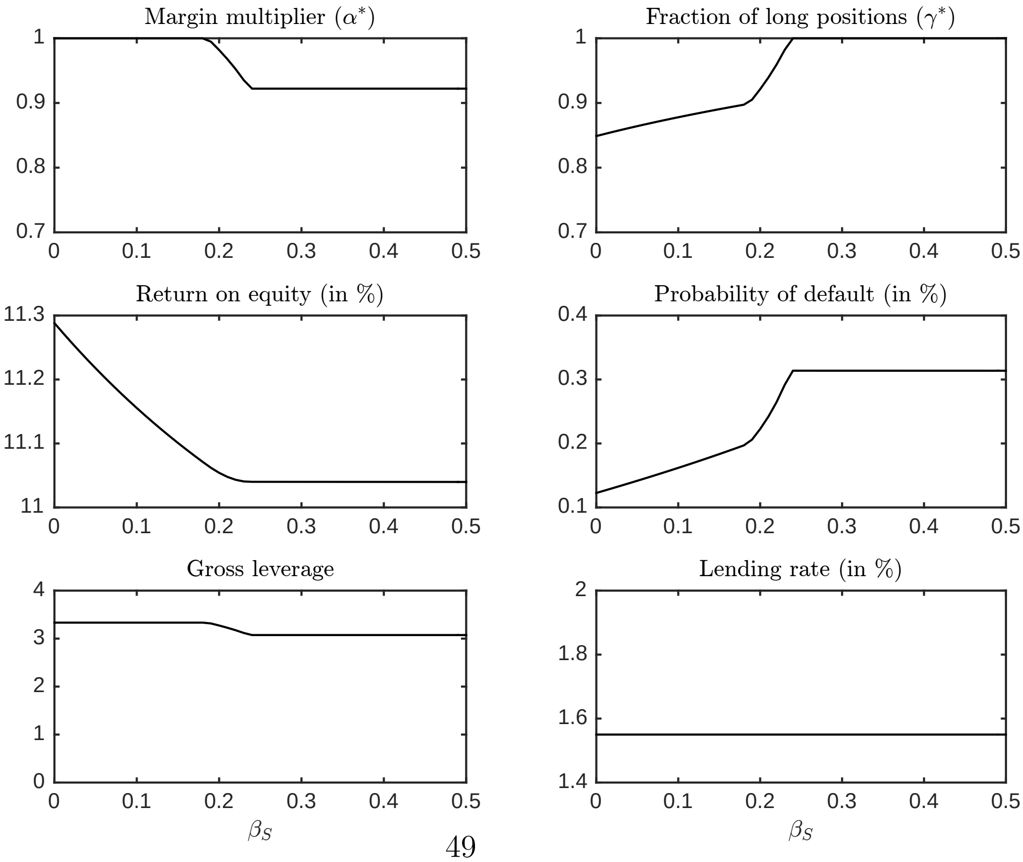

Figure 6: Hedge fund's decisions as a function of $\mu_L = \mu_S$

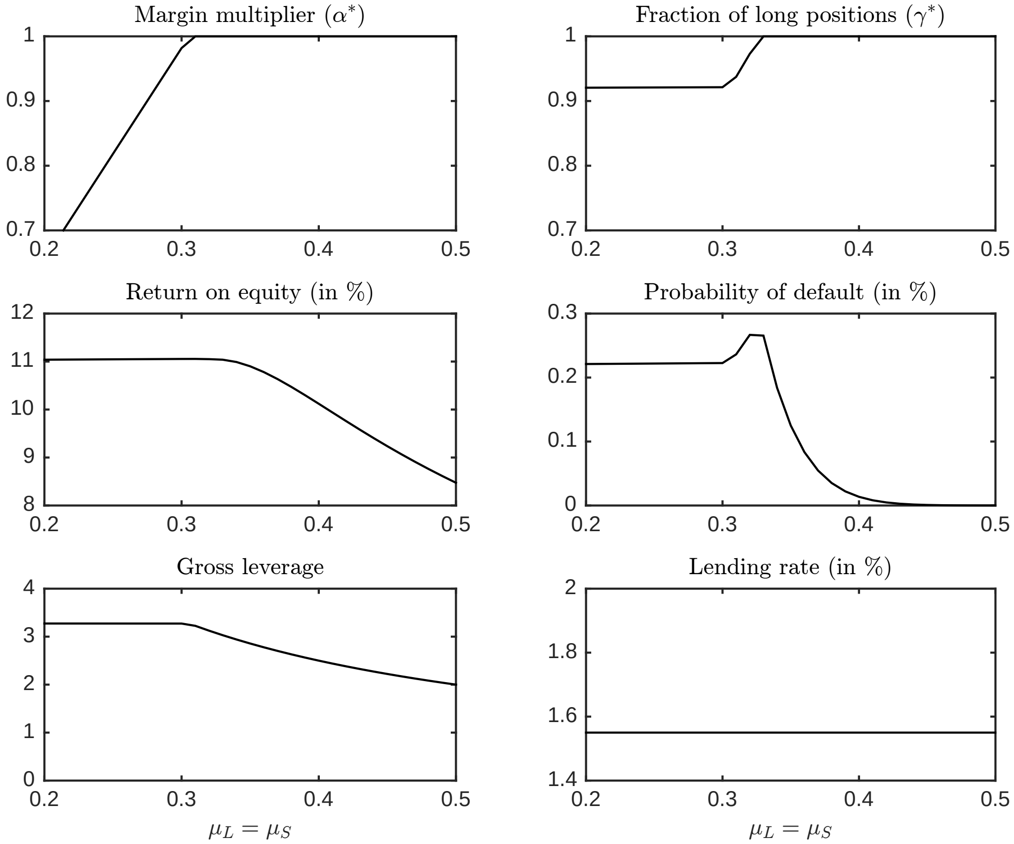

Figure 7: Hedge fund's decisions as a function of $m_L = m_S$

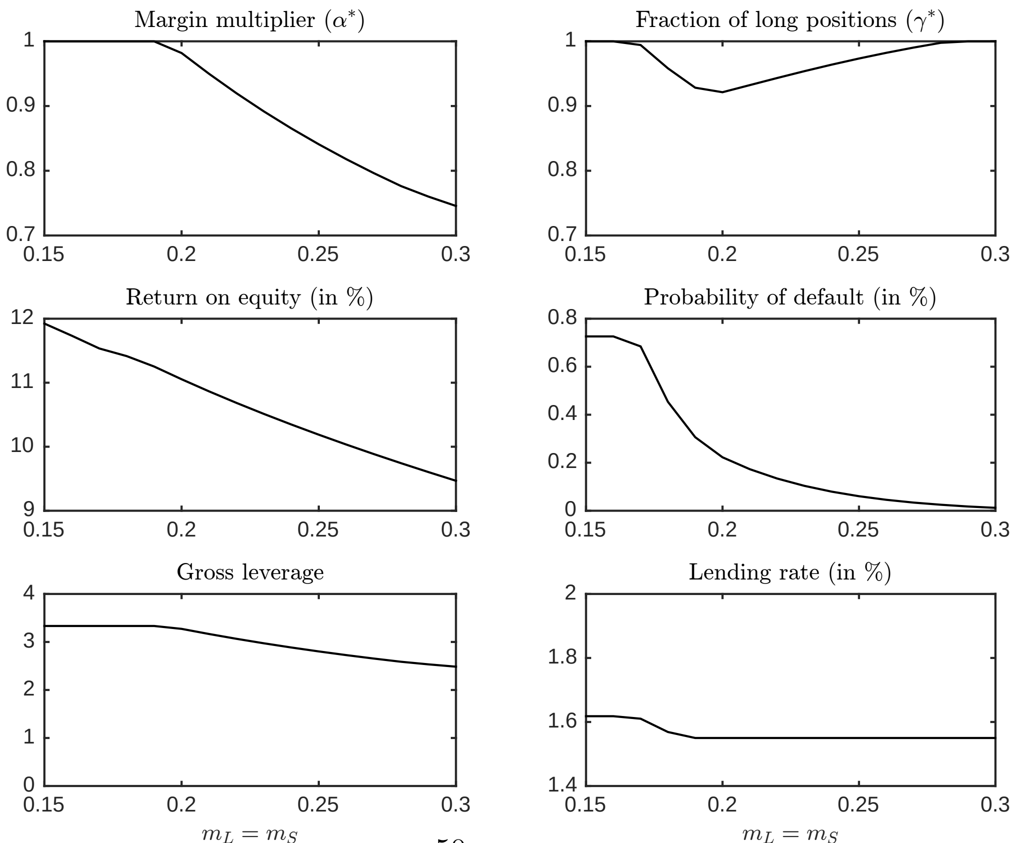

Figure 8: Hedge fund's decisions as a function of ϕ

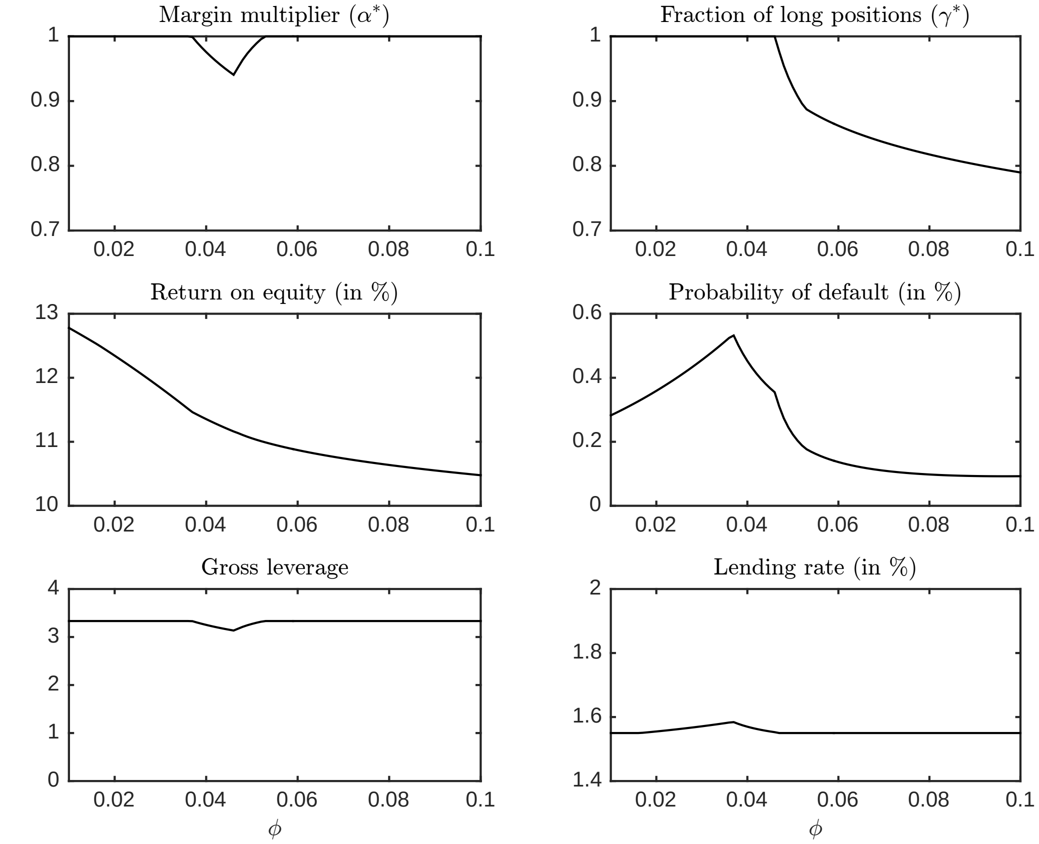

Figure 9: Hedge fund's decisions as a function of θ

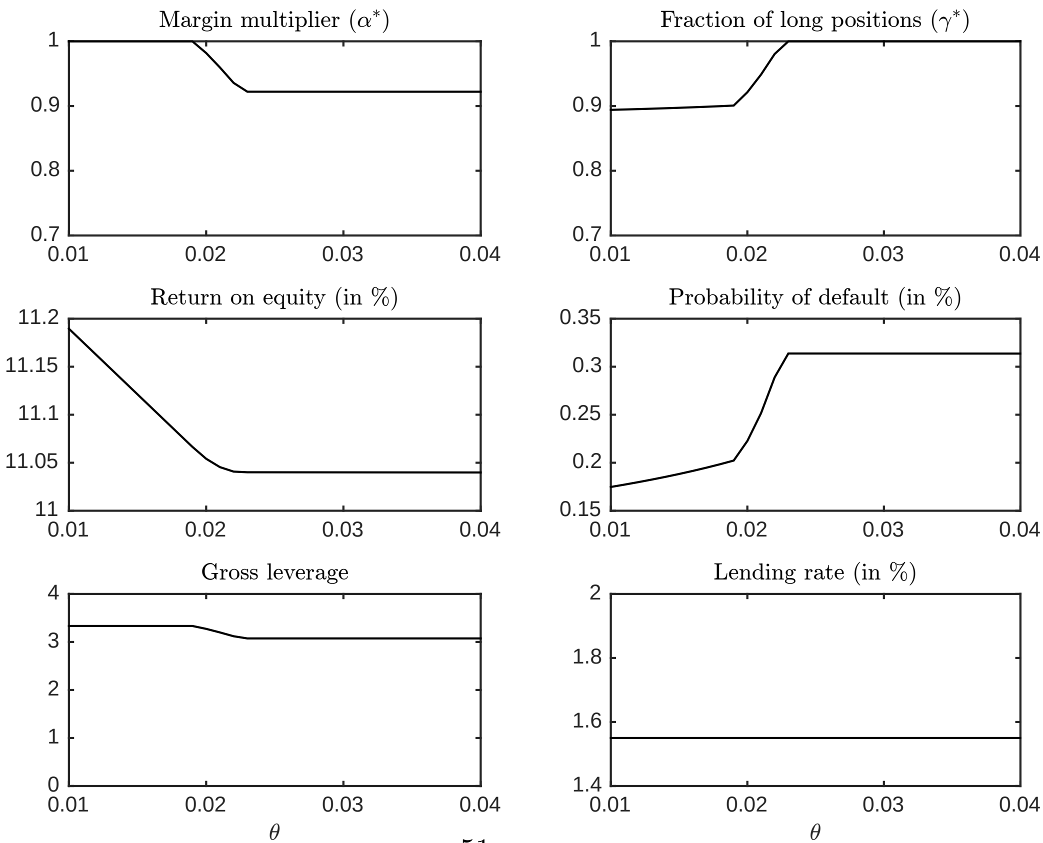

Figure 10: Value of the hedge fund and prime broker return on equity as a function of the market return