Portfolio Allocation with Budget and Risk Contribution Restrictions*

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Abstract

The standard mean-variance approach can imply extreme weights in some assets in the optimal allocation and a lack of stability over time. To improve the robustness of the portfolio allocation, but also to better control the portfolio turnover and account for transaction costs, it is proposed to introduce additional constraints on the risk contributions of the assets to the total portfolio risk. This approach is typically followed by the recent literature on risk parity, or equally weighted risk contribution portfolios. Our paper extends this literature in three directions: i) by considering other risk criteria than the variance, such as the Value-at-Risk (VaR), or the Expected Shortfall, ii) by weakening the risk contribution restrictions, and iii) by managing appropriately the systematic and idiosyncratic components of the portfolio risk.

Keywords: Asset Allocation, Portfolio Turnover, Risk Diversification, Minimum Variance Portfolio, Risk Parity Portfolio, Systematic Risk, Euler Allocation, Hedge Fund.

JEL classification: G12, C23.

1 Introduction

The gap between theory and practice is well illustrated by the example of portfolio management since Markowitz (1952) introduced the mean-variance framework. The resolution of the allocation problem by a simple quadratic optimization is the main advantage of the mean-variance approach. However, in practice this approach is implemented by replacing the theoretical mean and variance by their historical counterparts, and the associated estimated mean-variance portfolios have several drawbacks: they are very sensitive to errors in the estimates of the mean and variance inputs [see e.g. Chopra (1993), Chopra and Ziemba (1993)], the resolution of a large-scale quadratic optimization problem is not straightforward [see e.g. Konno and Hiroaki (1991)], and dominant factor in the covariance matrix results in extreme weights in optimal portfolios [see e.g. Green and Hollifield (1992)]. Finally the portfolio allocations are very erratic over time, which imply significant transaction costs or liquidity risks. These drawbacks are even more pronounced when the portfolio is based on a large number of assets.

These difficulties are due mainly to the sensitivity of the mean-variance efficient portfolio allocation to the smallest eigenvalues of the volatility matrix and to the poor accuracy of the inverse volatility matrix with the standard estimation methods. The literature has proposed different ways to get more robust portfolio allocations, as the potential cost of a loss of efficiency. First, some robust estimation methods have been introduced, following results known in statistics[1]. Typical of such approaches are the shrinkages of the estimated volatility matrix, which admit Bayesian interpretation [Garlappi, Uppal, Wang (2007), Goldfarb, Iyengar (2003), Ledoit, Wolf (2004)], the l_1- or l_2- penalizations introduced in the empirical optimization problem [see e.g. Broadie et al. (2008), DeMiguel et al. (2009)a, Fan et al. (2012)a], or the refresh time subsample approach with far more percentage of data used for any given pair of assets than for all the assets of the portfolio [Barndorff-Nielsen et al. (2008)].

Robustness can also be achieved by introducing restrictions in the empirical optimization problem even if these restrictions are not required by Financial Theory. These constraints have often

[1] It is well-known that the standard OLS estimator in a regression model $y = Xb + u$ is not very robust. The expression of the OLS estimator: $\hat{b} = (X'X)^{-1}X'y$ includes the inversion of the design matrix $X'X$, and this inversion is not very accurate when the explanatory variables are quasi-colinear. This lack of robustness is solved, either by considering Bayesian estimators, or by introducing l_2- penalizations, or by constraining the parameters.

simple interpretations. They can be shortselling restrictions [Frost, Savarino (1988), Chopra (1993) Jagannathan, Ma (2003)], gross exposure constraints [Fan et al. (2012)a], at the limit ”fully diversified” portfolios in terms of either budget allocations [see Elton, Gruber (1977), Duchin, Levy (2009), DeMiguel et al. (2009)b, Kritzman et al. (2010), Beleznay et al. (2012)] or contributions to total risk [see e.g. Martellini (2008), Choueifaty and Coignard (2008), Maillard et al. (2010)].

The idea of imposing additional diversification constraints is now commonly used in the asset management industry, and more enhanced strategies are grouped under the risk parity denomination. Risk parity is a general term for all investment techniques that attempt to take equal risk in the different underlyings of a portfolio. Let us consider the case of a multi-asset classes portfolio. A standard institutional allocation of 60% dollar exposure on stocks and 40% dollar exposure on bonds is not well diversified. Indeed, more than 90% of the portfolio volatility is due to the stocks exposure, since stocks are much more volatile than bonds. On the contrary, risk parity portfolios allocate 50% of risk on both stocks and bonds. This might result in 10% of dollar exposure to stocks, 90% to bonds. The historical return of such a portfolio might be something like 50% of the historical return of the 60% stock/40% bond portfolio, but with 25% of the risk. Risk parity portfolios must be levered up two times to get the same expected return as the initial portfolio, but with half the risk. In this case, the 10% stock/90% bond portfolio is more diversified than the 60% stock/40% bond initial one. Risk parity implementations differ considerably in practice. Asset classes, risk definition, risk forecasting methods and risk exposures calculation can be different from one manager to another one. Thus, risk parity is more a conceptual approach rather than a specific system, and it is in general difficult to compare the different approaches.

Many criticisms are made against the risk parity approaches. In general, the risk of a given portfolio is measured by its volatility. This justifies the most important criticism of risk parity portfolios [see e.g. Inker (2010)]. Indeed, risk should not be measured by standard deviation, but by the potential loss of capital. Once the potential loss of capital for each asset class has been estimated, together with the cross-correlations between these drawdown periods, the most efficient portfolio can be determined. Moreover, risk parity approaches increase portfolio dispersion by construction, in increasing the small cap weights. This is good in terms of risk (volatility), but this can create liquidity issues: we have to dynamically rebalance an equity portfolio with a bigger liquidity exposure on small caps. This point is not treated by the usual risk parity implementations.

We develop in this paper a new implementation of the risk parity principle that circumvents the usual limitations of the current implemented ones. Our contribution is threefold. First, we use a more appropriate risk measure that takes into account extreme risks. Second, we consider separately the systematic and idiosyncratic components of the portfolio risk. Third, we discuss the implied portfolio liquidity risk. The paper is organized as follows. In Section 2, we focus on the difference between the standard optimal portfolios and the associated risk parity portfolios. Section 3 considers asset returns with systematic and idiosyncratic components. Then we construct and compare different risk parity portfolios, when the parity is written on both types of components. Section 4 derives and compares optimal portfolios for different risk measures, especially the volatility, the Value-at-Risk and the Expected Shortfall. Section 5 presents empirical applications and Section 6 concludes.

2 Portfolio Allocation with Risk Contribution Restrictions

We review in this section basic results on portfolio and risk allocations to highlight the difference between the standard optimal portfolios and the portfolios with risk contribution restrictions. We denote by $y_1, ..., y_n$ the returns of n risky assets, Y the corresponding vector of returns, μ the vector of expected returns, Ω the associated volatility matrix, and w the portfolio allocation, satisfying the standardized budget constraint $w'e = 1$, with e being a n-dimensional vector of 1. We denote by $R(w)$ the scalar risk measure associated with allocation w. The risk measure depends on allocation w through the distribution of the portfolio return $w'Y$.

2.1 Minimum Risk Portfolios

Let us focus first on the risk minimization problem. We obtain the minimum risk allocation by solving the program:

$$w^* = \underset{w'e=1}{\operatorname{argmin}} R(w).$$

The optimization problem above is written under the standardized budget constraint $w'e = 1$. This possibility to standardize the budget constraint exists if the risk measure is homogeneous of degree

1, that is, if:

$$R(\lambda w) = \lambda R(w), \quad \text{say},$$

for any positive scalar λ. Indeed, the solution of an optimization problem such as:

$$w^*(\lambda) = \min_w R(w), \;\; s.t. \;\; w'e = 1/\lambda,$$

is equal to $w^*(\lambda) = \lambda w^*$. Thus, the solution with another budget restriction is deduced from the solution of the standardized optimization problem by an appropriate scaling.

2.2 Risk Contributions

The recent literature on risk measures focuses on the risk contribution of each asset to the total portfolio risk[2]. In this respect the risk contributions differ from the weights in portfolio allocations, since they also account for the effect of each individual asset on the total risk. Let us consider a global portfolio risk measured by $R(w)$. This total risk can be assigned to the different assets as:

$$R(w) = \sum_{i=1}^{n} R(w, w_i), \tag{2.1}$$

where $R(w, w_i)$ denotes the risk contribution of asset i to the risk of the whole portfolio. If the risk measure is homogenous of degree 1, we get the Euler formula:

$$R(w) = \sum_{i=1}^{n} w_i \frac{\partial R(w)}{\partial w_i}.$$

The Euler formula has an interpretation in terms of marginal contribution to global risk w.r.t. a change of scale in the portfolio allocation[3]. This explains why it is often proposed in the literature to choose:

$$R(w, w_i) = w_i \frac{\partial R(w)}{\partial w_i}, \tag{2.2}$$

[2]The risk contribution is also called risk budget in the literature [see e.g. Chow, Kritzmann (2001), Lee, Lam (2001)].

[3]The Euler formula is obtained by differentiating the homogeneity condition $R(\lambda w) = \lambda R(w)$, with respect to λ. We get:

$$\sum_{i=1}^{n} w_i \frac{\partial R(\lambda w)}{\partial w_i} = R(w),$$

and deduce the Euler formula by setting $\lambda = 1$.

called the Euler allocation [Litterman (1996), p.28, Garman (1997), footnote 2, Qian (2006)]. The difference between the portfolio allocation and the risk contribution is clear from (2.2), and is captured by the marginal risk $\partial R(w)/\partial w_i$.

2.3 Portfolios with Risk Contribution Restrictions

The Euler decomposition can be used to construct portfolios with constraints on the risk contributions.

i) For instance, Risk Parity portfolios, or Equally Weighted Risk Contribution (ERC) portfolios have been considered in the literature [see e.g. Scherer (2007), Maillard et al. (2010), Asness et al. (2012)], and are gaining in popularity among practitioners [Asness (2010), Sullivan (2010), Dori et al. (2011)].

They are characterized by the conditions:

$$\begin{aligned} & w_i \frac{\partial R(w)}{\partial w_i} && \text{independent of } i, \text{ with } w'e = 1. \\ \Leftrightarrow\ & \frac{\partial R(w)}{\partial w}(w) &=\ & \lambda(w)\ \mathrm{vec}(1/w), \text{ for a multiplier } \lambda(w), \text{ such that } w'e = 1, \end{aligned} \tag{2.3}$$

where $\mathrm{vec}(1/w)$ is the vector with components $1/w_i$. The risk parity portfolios are diversified by risk unit, not by dollar. This notion of risk diversification depends on the selected risk measure (see Section 4).

ii) This practice can be extended by imposing the risk contributions to be proportional to some benchmarks π_i, $i = 1, ..., n$, which are not necessarily equal to $1/n$:

$$w_i \frac{\partial R(w)}{\partial w_i} = \lambda(w)\ diag(\pi_i)\ \mathrm{vec}(1/w), \tag{2.4}$$

where $diag(\pi_i)$ is the diagonal matrix with π_i $i = 1, \ldots, n$ as diagonal elements.

2.4 Risk Contribution Restrictions and Portfolio Turnover

It is possible to justify the introduction of restrictions (2.4) by the effect of trading costs. Let us assume that the investor's portfolio allocation at the beginning of the period is: $w_0 = (w_{0,1}, ..., w_{0,n})'$, and that the investor updates his portfolio to get the new allocation $w = (w_1, ..., w_n)'$. He will account for the risk $R(w)$ of the new allocation and for the trading costs when passing from w_0 to w. Under no short sale constraints: $w_{0,i} \geq 0, w_i \geq 0, \forall i$, the trading cost may be measured by:

$$c \sum_{i=1}^{n} w_{0,i} \ln \left(\frac{w_{0,i}}{w_i} \right).$$

Indeed, when the allocation adjustment is not too large, we have:

$$\begin{aligned} c \sum_{i=1}^{n} w_{0,i} \ln \left(\frac{w_{0,i}}{w_i} \right) &= -c \sum_{i=1}^{n} w_{0,i} \ln \left(1 + \frac{w_i - w_{0,i}}{w_{0,i}} \right) \\ &\approx -c \left[\sum_{i=1}^{n} w_{0,i} \frac{w_i - w_{0,i}}{w_{0,i}} - \frac{1}{2} \sum_{i=1}^{n} w_{0,i} \frac{(w_i - w_{0,i})^2}{w_{0,i}^2} \right] \\ &\approx \frac{c}{2} \sum_{i=1}^{n} \frac{(w_i - w_{0,i})^2}{w_{0,i}}, \end{aligned}$$

since the two portfolios satisfy the budget constraint: $e'w = e'w_0 = 1$.

This approximation has a direct interpretation in terms of transaction costs, in which the cost for trading asset i is proportional to $1/w_{0,i}$. This assumption on trading costs can find a justification if the initial allocation corresponds to a market portfolio. Assets with the highest market weights w_{oi} are also the most liquid ones, and then their trading is associated with low transaction cost. At the opposite, assets with the lowest market weights are less liquid and then trading these assets is expensive in terms of transaction costs. This cost for trading asset i is proportional to $(w_i - w_{0,i})^2$. Thus the implied market impact function for trading asset i is strictly convex.

The investor has to balance risk control and trading cost in his portfolio management. Thus, he can minimize a combination of both criteria, and choose:

$$w = \underset{w}{\operatorname{argmin}}\, R(w) + \lambda c \sum_{i=1}^{n} w_{0,i} \ln \left(\frac{w_{0,i}}{w_i} \right), \tag{2.5}$$

where $\lambda > 0$ is a smoothing parameter introduced to control the portfolio turnover. With $\lambda = 0$, the investment objective focuses on risk control. For high λ, the control is on the portfolio turnover, and the investment objective is to enhance the initial portfolio allocation in terms of risk control, but with a limited turnover.

The associated first-order condition is:

$$\frac{\partial R(w)}{\partial w_i} - \lambda\, c\, \frac{w_{0,i}}{w_i} = 0$$
$$\Leftrightarrow w_i \frac{\partial R(w)}{\partial w_i} = \lambda\, c\, w_{0,i}. \tag{2.6}$$

The risk contributions are proportional to the initial portfolio allocations: $\pi_i \propto w_{0,i}$. In this interpretation, there is no reason to look for risk parity portfolios[4]. Moreover, the choice of the levels of risk contributions π_i, $i = 1, ..., n$ depends on the current investor's portfolio. This approach is clearly suitable to advise investors that already have a well-defined portfolio, and want to enhance their risk management without generating a high portfolio turnover.

This solution is especially appealing in a multi-period framework. Indeed, in a myopic dynamic portfolio management, the sequence of optimization problems is:

$$w_t^* = \underset{w_t}{\operatorname{argmin}}\, R_t(w_t) + \lambda c_t \sum_{j=1}^{n} w_{t-1,j}^* \ln\left(\frac{w_{t-1,j}^*}{w_{t,j}}\right),$$

where the conditional risk measure $R_t(w_t)$ and the trading cost c_t depend on time. Then the risk contribution restrictions are proportional to $w_{t-1,i}^*$ and path dependent. In this dynamic framework, the λ parameter controls the speed of convergence of the current portfolio towards the time dependent minimum risk portfolio. In a stable risk environment, that is, if R_t does not depend on time, the optimal dynamic reallocation approaches the minimum risk portfolio in several steps instead of doing the reallocation at a single date. This point is especially appealing for big institutional investors that want to reallocate huge portfolios without destabilizing financial markets. This multi-period optimal reallocation approach is also appealing when managing portfolios of illiquid assets.

3 Portfolio Allocation with Systematic and Idiosyncratic Risk Contribution Restrictions

In this section, we consider portfolio allocations constructed to monitor the systematic and idiosyncratic components of the portfolio return. This is done by imposing the risk contribution

[4] The usual risk parity portfolio discussed in Maillard et al. (2010) is the optimal reallocation decision of an investor holding an equally weighted portfolio and who wants to control the portfolio volatility.

restrictions on these two components of the total risk. We consider factor models to discuss the effects of the systematic and idiosyncratic components of the risk.

3.1 Systematic and Idiosyncratic Risks

Let us assume that the asset returns follow the one-factor model:

$$y_i = \beta_i \, f + \sigma_i \, u_i, \qquad i = 1, ..., n, \tag{3.1}$$

where f is the common (or systematic) factor, β_i is the factor loading of asset i on factor f, and u_i is the idiosyncratic (or specific) component, independent of the factor. We assume that the idiosyncratic terms are mutually independent[5], with unconditional zero mean and unit variance. We get the following decomposition of the return covariance matrix:

$$\Omega = \beta\beta' \sigma_f^2 + \Sigma, \tag{3.2}$$

where $\Sigma = Vu = diag(\sigma^2)$, σ_f^2 is the variance of the factor and β the vector of factor loadings. The portfolio return can be decomposed accordingly into a systematic and an idiosyncratic component as:

$$w'Y = \left(\sum_{i=1}^{n} w_i \beta_i \right) f + \sum_{i=1}^{n} w_i \sigma_i u_i. \tag{3.3}$$

The effects of the systematic and idiosyncratic components can be analyzed for both risk contributions and portfolio allocations.

3.2 Systematic and Idiosyncratic Risk Contributions

The decomposition principle (2.1) can be applied to disentangle the systematic and idiosyncratic components of the risk as follows:

$$R(w, w_i) = R_s(w, w_i) + R_u(w, w_i),$$

where $R_s(w, w_i)$ [resp. $R_u(w, w_i)$] denotes the systematic (resp. idiosyncratic) risk contribution of asset i on the total systematic (resp. idiosyncratic) component of the risk. The risk decompositions

[5]Any residual dependence might be captured by means of additional common factors. This would lead to a multi-factor model. We consider the one-factor model for expository purpose.

above can be aggregated to get a decomposition of the total risk as:

$$R(w) = R_s(w) + R_u(w),$$

with $R_s(w) = \sum_{i=1}^n R_s(w, w_i)$ and $R_u(w) = \sum_{i=1}^n R_u(w, w_i)$. These decompositions are summarized in Table 1. This table shows how to pass from the assets $i = 1, ..., n$ tradable on the market, to the virtual assets f and $(u_1, ..., u_n) = u$, which are not directly tradable, that is, how to transform the decomposition of the total risk with respect to basic assets $i = 1, ..., n$ to a decomposition with respect to virtual assets. This is done by constructing an appropriate bivariate table, and summing up by columns instead of summing up by rows [see Gourieroux, Monfort (2012)].

Assets	Systematic factor	Idiosyncratic error terms	Total
1			$R(w, w_1)$
$\vdots$			$\vdots$
i	$R_s(w, w_i)$	$R_u(w, w_i)$	$R(w, w_i)$
$\vdots$			$\vdots$
n			$R(w, w_n)$
Total	$R_s(w)$	$R_u(w)$	$R(w)$

Table 1: Decomposition of the Global Risk Measure

How to derive this thinner risk decomposition in practice, while keeping an interpretation in terms of Euler decomposition? Let us consider the virtual portfolio with allocation $w_{i,s}$ in the systematic component and $w_{i,u}$ in the idiosyncratic one. Thus the associated portfolio return becomes:

$$\left(\sum_{i=1}^n w_{i,s} \beta_i \right) f + \sum_{i=1}^n w_{i,s} \sigma_i u_i.$$

This portfolio invests $w_{i,s}$ in $\beta_i f$, $w_{i,u}$ in $\sigma_i u_i$. If we denote $\tilde{w}$ (resp. $\tilde{Y}$) the components $w_{i,s}, w_{i,u}$ (resp. the virtual basic assets $\beta_i f$, $i = 1, ..., n$, $\sigma_i u_i$, $i = 1, ..., n$), the risk measure of this virtual portfolio can be written as: $\tilde{R}(w_s, w_u)$, where $\tilde{R}(w, w) = R(w)$. The extended risk measure $\tilde{R}$ is also homogenous of degree 1. Thus we can applied the Euler formula to $\tilde{R}$ and get:

$$\tilde{R}(w_s, w_u) = \sum_{i=1}^n w_{i,s} \frac{\partial \tilde{R}}{\partial w_{i,s}}(w_s, w_u) + \sum_{i=1}^n w_{i,u} \frac{\partial \tilde{R}}{\partial w_{i,u}}(w_s, w_u).$$

Then, for $w_s = w_u = w$, we deduce the thinner decomposition:

$$\tilde{R}(w) = \sum_{i=1}^{n} w_i \frac{\partial \tilde{R}}{\partial w_{i,s}}(w,w) + \sum_{i=1}^{n} w_i \frac{\partial \tilde{R}}{\partial w_{i,u}}(w,w),$$

and can define: $R_s(w, w_i) = w_i \frac{\partial \tilde{R}}{\partial w_{i,s}}(w,w)$, $R_u(w, w_i) = w_i \frac{\partial \tilde{R}}{\partial w_{i,u}}(w,w)$. Finally, the risk measure $R(w)$ is also function of parameters β_i, σ_i, $i = 1, ..., n$, involved in the factor model and we get:

$$\frac{\partial \tilde{R}}{\partial w_{i,s}}(w,w) = \frac{\partial R}{\partial \beta_i}(w),\ \frac{\partial \tilde{R}}{\partial w_{i,u}}(w,w) = \frac{\partial R}{\partial \sigma_i}(w), \tag{3.4}$$

in which the dependence of R with respect to β_i, σ_i is not explicitely written for expository purpose. Thus, we get a decomposition, which highlights the effects on the total portfolio risk of shocks on the factor and/or on the idiosyncratic term .

3.3 Portfolios with Constraints on the Systematic Risk Contribution

In the standard portfolios with risk contribution restrictions, the constraints are written on the basic assets. The approach can be extended by considering risk contributions written on the systematic and unsystematic components of the portfolio. Let us denote by π the proportion of risk allowed for the systematic component and $1 - \pi$ the proportion for the unsystematic component. The optimization problem becomes:

$$\min_{w} R(w) \tag{3.5}$$
$$s.t.\ R_s(w) = \pi R(w)\ e'w = 1.$$

In the optimization above, the restriction on the systematic risk contribution is strict. It can be weakened by considering the following optimization problem:

$$w(\delta, \pi) = \underset{w'e=1}{\text{argmin}}\, R^2(w) + \delta \left[(1-\pi)R_s(w) - \pi R_u(w)\right]^2, \tag{3.6}$$

where $\delta \in (0, \infty)$ is a smoothing parameter. In the limiting case $\delta = \infty$, we get the optimization (3.5) with strict constraints. When $\delta = 0$, we get the minimum risk portfolio.

As in Section 2.4, we can justify the introduction of these risk contribution restrictions by the effect of trading costs, both on individual assets and on the factors, when derivative instruments allows investors to directly trade on the virtual factor asset. This is the case for equity investing, where the factor is usually the market portfolio.

4 Illustrations with Different Risk Measures

This section provides the closed form expressions of the minimum risk portfolios and the risk contributions for different risk measures, such as the volatility, the Value-at-Risk and the Expected Shortfall.

4.1 The Volatility Risk Measure

When the risk is measured by the volatility, we get:

$$R(w) = (w'\Omega w)^{\frac{1}{2}}.$$

i) Minimun variance portfolio

Let us assume that the set of assets does not include the riskfree asset, or equivalently that the volatility matrix Ω is invertible. For the volatility risk measure, we get the minimum-variance portfolio [see Markowitz (1952)]. The optimal allocation has the closed form expression:

$$w^* = \frac{\Omega^{-1}e}{e'\Omega^{-1}e}.$$

ii) Basic risk contributions

We have:

$$\frac{\partial R(w)}{\partial w} = \frac{\Omega w}{R(w)},$$

and the risk contributions are:

$$R(w, w_i) = \frac{w_i}{R(w)} \sum_{j=1}^{n} \Omega_{i,j}\, w_j,$$

where $\Omega_{i,j}$ is the generic element of Ω. We get: $R(w, w_i) = Cov(w_i y_i, w'Y)/V(w'Y)$, that is the beta coefficient of the part of the portfolio invested in asset i with respect to the total portfolio.

iii) Risk parity portfolio

The first-order conditions are:

$$\Omega w = \lambda(w)\,\text{vec}(1/w), \text{ for a given } \lambda(w). \tag{4.1}$$

• If $\Omega = diag(\sigma^2)$ is a diagonal matrix, the conditions become: $w_i^2\sigma_i^2 = \lambda(w)$ independent of i. Therefore, w_i has to be proportional to $1/\sigma_i$ (up to the sign). This portfolio differs from the minimum-variance portfolio in which w_i is proportional to $1/\sigma_i^2$. With the volatility risk measure, the risk parity portfolio overweights safer assets relative to their weights on the mininum variance portfolio. This interpretation can fail if the Ω matrix is not diagonal.

• When Ω is not diagonal, the nonlinear system (4.1) has to be solved numerically. Moreover, the nonlinearity dimension is equal to the total number of assets in the portfolio. This could imply numerical difficulties in the resolution of this allocation problem. Maillard *et al.* (2010) noted that the solution of this system under short sale restrictions is also the solution of the problem:

$$w = \underset{w}{\text{argmin}}(w'\Omega w)^{\frac{1}{2}},$$

under the constraints $\sum_{i=1}^{n} ln(w_i) \geq c$, c being an arbitrary constant. The budget constraint $w'e = 1$ is satisfied in a second step by renormalizing the portfolio weights.

iv) Systematic and idiosyncratic risk contributions

Let us consider the single factor model. We have: $R(w) = [w'(\beta\beta'\sigma_f^2 + \Sigma)w]^{1/2}$ and the Euler risk contributions can be written as:

$$R(w, w_i) = \frac{w_i}{R(w)}\left[\beta_i\, w'\, \beta\, \sigma_f^2 + w_i\, \sigma_i^2\right] = R_s(w, w_i) + R_u(w, w_i),$$

where $R_s(w, w_i)$ is the risk contribution associated with the factor capturing the systematic component of asset i and $R_u(w, w_i)$ is the idiosyncratic risk contribution of asset i:

$$R_s(w, w_i) = w_i\, \beta_i \frac{w'\beta\, \sigma_f^2}{R(w)}, \qquad R_u(w, w_i) = w_i \frac{w_i\, \sigma_i^2}{R(w)}.$$

The expression of component $R_s(w, w_i)$ shows the quantity $w_i\beta_i$ invested in the systematic factor f, and the risk contribution $\dfrac{w'\beta\ \sigma_f^2}{R(w)}$ of a unit invested in f. By adding the decompositions per asset, we get the decomposition of the total portfolio risk as:

$$R(w) = R_s(w) + R_u(w), \text{ with } R_s(w) = (w'\beta)^2 \frac{\sigma_f^2}{R(w)} \text{ and } R_u(w) = \frac{w'\Sigma w}{R(w)}.$$

In this framework we get the standard variance decomposition equation.

v) Minimum variance portfolio and systemic risk contribution

Let us decompose the minimum-variance allocation to disentangle the effects of systematic and unsystematic components. This portfolio is given by:

$$w^* = \frac{\Omega^{-1}e}{e'\Omega^{-1}e} = \frac{(\beta\beta'\sigma_f^2 + \Sigma)^{-1}e}{e'(\beta\beta'\sigma_f^2 + \Sigma)^{-1}e}.$$

The inverse $(\beta\beta'\sigma_f^2 + \Sigma)^{-1}$ admits the explicit expression:

$$(\beta\beta'\sigma_f^2 + \Sigma)^{-1} = \Sigma^{-1} - \frac{\sigma_f^2\ \Sigma^{-1}\beta\beta'\Sigma^{-1}}{1 + \sigma_f^2\ \beta'\Sigma^{-1}\beta}.$$

We deduce that:

$$w^* = \frac{\Sigma^{-1}e + \sigma_f^2[\beta'\Sigma^{-1}\beta\Sigma^{-1}e - \beta'\Sigma^{-1}e\Sigma^{-1}\beta]}{e'\Sigma^{-1}e + \sigma_f^2[\beta'\Sigma^{-1}\beta e'\Sigma^{-1}e - (\beta'\Sigma^{-1}e)^2]}. \tag{4.2}$$

Thus, w^* is a weighted average of the optimal allocation in the idiosyncratic virtual assets, i.e. $w_k^* = \frac{\Sigma^{-1}e}{e'\Sigma^{-1}e}$, and of the optimal allocation in the systematic virtual asset, i.e.

$$w_s^* = \frac{\beta'\Sigma^{-1}\beta\Sigma^{-1} - \beta'\Sigma^{-1}e\Sigma^{-1}\beta}{\beta'\Sigma^{-1}\beta e'\Sigma^{-1} - (\beta'\Sigma^{-1}e)^2}.$$

Symmetrically, we get a decomposition of the total risk of the minimum variance portfolio into its systematic and unsystematic risk contributions. We get:

$$R_s(w^*) = \frac{(e'\Omega^{-1}\beta)^2\sigma_f^2}{(e'\Omega^{-1}e)^{3/2}}, \qquad R_u(w^*) = \frac{e'\Omega^{-1}\Sigma\Omega^{-1}e}{(e'\Omega^{-1}e)^{3/2}}. \tag{4.3}$$

4.2 The α-VaR Risk Measure

Other risk measures than the (conditional) variance can be considered, if we want to focus on extreme risks. The introduction of the VaR, or of the Expected Shortfall corresponds to the safety first criterion initially introduced by Roy (1952).

The α-VaR risk measure is defined by:

$$R(w) = -q_\alpha(w'Y),$$

where q_α is the α-quantile of the distribution of the portfolio return. More precisely, the α-VaR is defined by the condition: $P\left[w'Y < q_\alpha(w'Y)\right] = \alpha$.

i) Minimum α-VaR portfolio

Let us first consider the Gaussian case before discussing the general framework.

• Let us assume that the set of basic assets does not include the riskfree asset and consider the allocation minimizing the α-VaR in a Gaussian framework. When the vector of returns is Gaussian with mean μ and variance-covariance Ω, the optimal allocation minimizes:

$$-q_\alpha(w'Y) = -w'\mu - q_\alpha w'\Omega w,$$

where q_α denotes the α-quantile[6] of the standard Gaussian distribution under the budget restriction $w'e = 1$. The minimum α-VaR portfolio allocation is then given by:

$$w^* = \frac{\Omega^{-1}e}{e'\Omega^{-1}e} + \frac{1}{2q_\alpha}\Omega^{-1}\left[\mu - \frac{e'\Omega^{-1}\mu}{e'\Omega^{-1}e}e\right].$$

This formula highlights the key role of the minimum variance portfolio as the benchmark portfolio for a very risk averse investor (when $\alpha \to 0$ and $q_\alpha \to \infty$), but also the importance of the excess expected returns.

• In the general case, the returns are not necessarily Gaussian and the minimum α-VaR portfolio is the solution of the system of equations:

[6] Since α is small, q_α is negative. Thus, the α-VaR is an increasing function of the variance of the portfolio return and a decreasing function of its expected return.

$$\frac{\partial q_\alpha(w'Y)}{\partial w_i} = \lambda(w), i = 1, ..., n, \tag{4.4}$$

where the Lagrange multiplier $\lambda(w)$ is fixed by the budget restriction $w'e = 1$. The derivative of the α-VaR is equal to [Gourieroux, Laurent, Scaillet (2000), Hallerbach (2003)]:

$$\frac{\partial q_\alpha}{\partial w_i}(w'Y) = E\left[y_i | w'Y = q_\alpha(w'Y)\right], i = 1, ..., n. \tag{4.5}$$

This derivative has no closed form expression in general and the minimum α-VaR allocation has to be computed numerically.

ii) Basic risk contributions

When the risk is measured by the α-VaR, we get the following decomposition formula of the global conditional quantile [see Gourieroux, Monfort (2012)]:

$$q_\alpha(w'Y) = w' \frac{\partial q_\alpha(w'Y)}{\partial w} = w' E\left[Y | w'Y = q_\alpha(w'Y)\right], \tag{4.6}$$

and

$$R(w, w_i) = E\left[w_i y_i | w'Y = q_\alpha(w'Y)\right]. \tag{4.7}$$

It measures the part of the expected loss due to asset i when the total portfolio is in distress.

iii) Risk parity portfolios

The risk parity restrictions become:

$$E\left[w' y_i | w'Y = q_\alpha(w'Y)\right] = \lambda(w), \text{ independent of asset } i. \tag{4.8}$$

These conditions have usually to be solved numerically. In the Gaussian case, the nonlinear system of equations becomes:

$$\mu + 2\, q_\alpha\, \Omega\, w = \lambda(w)\ \text{vec}(1/w). \tag{4.9}$$

Thus, with an α-VaR risk measure, the risk parity portfolios depend on both the volatility and the expected returns.

iv) Systematic and idiosyncratic risk components

In the α-VaR case, the marginal effect of a change of weight of asset i can be decomposed by Equation (3.4) as:

$$\frac{\partial q_\alpha(w'Y)}{\partial w_i} = \beta_i \frac{\partial q_\alpha(w'Y)}{\partial \beta_i} + \sigma_i \frac{\partial q_\alpha(w'Y)}{\partial \sigma_i}. \tag{4.10}$$

The Euler components associated with systematic and idiosyncratic risks can be explicited as follows:

$$\frac{\partial q_\alpha(w'Y)}{\partial \beta} = E\left[f|w'Y = q_\alpha(w'Y)\right], \; \frac{\partial q_\alpha(w'Y)}{\partial \sigma_i} = E\left[u_i|w'Y = q_\alpha(w'Y)\right].$$

In the linear factor model, the general decomposition (4.7) becomes:

$$R_s(w, w_i) = w_i\beta_i E\left[f|w'Y = q_\alpha(w'Y)\right], \; R_u(w, w_i) = w_i\sigma_i E\left[u_i|w'Y = q_\alpha(w'Y)\right], \tag{4.11}$$

and the decomposition of the total portfolio risk is:

$$R(w) = R_s(w) + R_u(w),$$

with

$$R_s(w) = w'\beta \; E\left[f|w'Y = q_\alpha(w'Y)\right], \;\; R_u(w) = \sum_{i=1}^{n} w_i\sigma_i E\left[u_i|w'Y = q_\alpha(w'Y)\right]. \tag{4.12}$$

4.3 Distortion Risk Measure

A distortion risk measure is a weighted function of the VaRs associated with the different risk levels. It can be written as:

$$R(w) = \int VaR_\alpha(w) dH(\alpha) = -\int q_\alpha(w) dH(\alpha),$$

where H is a given distortion measure on $(0, 1)$, that is, an increasing concave function. The Expected Shortfall is obtained when H is the cumulative distribution function of the uniform distribution on the interval $[0, \alpha]$ [see e.g. Wang (2000), Acerbi (2002), Acerbi, Tasche (2002)].

i) Minimum DRM portfolio

Let us still consider successively the Gaussian and general cases.

• For distortion risk measures and Gaussian returns, the criterion becomes:

$$DRM_\alpha(w'Y) = -\int q_\alpha(w'Y)dH(\alpha) = -w'\mu - \left[\int q_\alpha dH(\alpha)\right] w'\Omega w,$$

where q_α is the $\alpha-$quantile of the standard Gaussian distribution. Thus, the minimum DRM allocation is:

$$w^*(H) = \frac{\Omega^{-1}e}{e'\Omega^{-1}e} + \frac{1}{2DRM_\alpha}\Omega^{-1}\left[\mu - \frac{e'\Omega^{-1}\mu}{e'\Omega^{-1}e}e\right],$$

where $DRM_\alpha = \int q_\alpha dH(\alpha)$ is the distortion risk measure of the standard Gaussian distribution. The set of optimal portfolios when α varies is the same for any choice of the distortion measure, and is equal to the set of the mean-variance portfolios in absence of riskfree asset. We get a two funds separation theorem, where this set is generated by the minimum variance portfolio $\frac{\Omega^{-1}e}{e'\Omega^{-1}e}$ and the long-short dollar neutral portfolio $\Omega^{-1}\left[\mu - \frac{e'\Omega^{-1}\mu}{e'\Omega^{-1}e}e\right]$.

• In non Gaussian cases, the optimal allocations have no closed form expression and have to be derived numerically. The minimum DRM portfolios solve the first-order condition [see Gourieroux *et al.* (2000)]:

$$(1 - q_\alpha(w'Y))\, E\left[Y|w'Y = q_\alpha(w'Y)\right] = 0,$$

and differ generally from the mean-variance portfolios.

ii) Basic risk contributions

Let us for instance consider the Expected Shortfall ES_α. By definition we have:

$$ES_\alpha(w'Y) = w'E\left[Y|w'Y > q_\alpha(w'Y)\right], \tag{4.13}$$

with risk contribution [Tasche (2000)]: $R(w, w_i) = E\left[w_i y_i|w'Y > q_\alpha(w'Y)\right]$. Thus, risk decompositions (4.6) and (4.13) differ by their conditioning set. These conditioning sets correspond to different definitions of portfolio distress.

iii) Risk parity portfolios

As for the α-VaR framework, we do not have usually a closed form expression of the risk parity portfolios.

iv) Systematic and idiosyncratic risk components

The ES risk measure provides the following risk contributions [7]:

$$R_s(w) = w'\beta\, E\,[f|w'Y > q_\alpha(w'Y)],\;\; R_u(w) = \sum_{i=1}^{n} w_i\sigma_i E\,[u_i|w'Y > q_\alpha(w'Y)]. \qquad (4.14)$$

with simply the change in the definition of portfolio distress.

5 Application

We apply in this section the portfolio managements above to futures on commodities.

5.1 The assets

We consider futures contracts on physical commodities. These assets are split into five sectors as described in Table 2.

Table 2 : The Commodities

Energy	Grands & Seeds	Softs	Live stock	Metals
brent crudeoil*	corn*	cocoa	lean hogs*	copper*
heating oil*	rice	coffee*	live cattle*	gold*
light crudeoil*	soybean oil*	cotton*		palladium
natural gas*	soybeans*	orange juice		platinum
	wheat*	sugar*		silver*

The prices are daily closing prices from 14 May 1990 up to 24 September 2012 and are all denominated in US$, even for metals traded in London. The physical commodity prices include

[7] Called Marginal Expected Shortfall (MES) in the recent literature on systemic risk [see e.g. Acharya et al. (2010), Brownlees, Engle (2011)].

the storage and transportation costs. The returns are adjusted by rolling the futures position in order to avoid the delivery process.

5.2 Summary statistics

We provide in Table 3 summary statistics of the historical distribution of returns, that are the historical mean, volatility, skewness and kurtosis, with the historical Value-at-Risk for risk levels 1%, 5%, 95%, 99%. We also display the historical betas of each asset return with respect to the Dow Jones-UBS (DJUBS) commodity index.

[Insert Table 3 : Summary Statistics of Asset Returns].

We observe rather symmetric distributions, except for commodity ”brent crudeoil”, which is left skewed, and for ”cotton”, which is right skewed. All distributions feature tails fatter than Gaussian tail with kurtosis up to 30-40 for ”brent crudeoil” and ”cotton”.

The betas are all nonnegative and some returns are very sensitive to changes in the market index such as ”brent crudeoil” and ”soybeans”. These large values do not only reflect the composition of the DJUBS index. Indeed this index includes currently 20 physical commodities for 7 sectors. Thus, several commodities in Table 2 are not included in the index. The commodities included in the index are marked with a ”*” in Table 2. Moreover, if the weights of included assets are fixed according to their global economic significance and market liquidity, they are capped. No commodity can compose more than 15% of the index and no sector more than 33%. For instance coca, coffee and cotton have similar weights, but cotton has a much higher beta than the two other commodities.

This analysis can be completed by considering the historical bivariate distributions for any pair of assets. We provide in Table 4 the historical correlations between asset returns and in Figure 1 the scatterplots of the bivariate distributions for the Grains & Seeds sector, the one-dimensional distribution being displayed on the diagonals.

[Insert Table 4 : Historical Correlation Matrix]

[Insert Figure 1 : Scatterplots of One and Bi-dimensional Distributions Sector: Grains & Seeds]

Table 3 and Figure 1 provide similar information on the pairwise links between asset returns. We get high correlations especially between ”oil” commodities and within the ”Live Stock” sector. Let us now focus on the sector ” Grains & Seeds” and on Figure 1, where the scatterplots are easy to interpret. We observe some independence between returns for a significant number of pairs, but also strong positive links for pairs of substituable commodities such as ”soybeans” and ”soybean oil”, or for ”corn” and ”wheat”. We even observe multiregimes of dependence for ”soybean” and ”wheat”, where the scatterplot shows two different regression lines.

5.3 Return dynamics

For expository purpose, it is not possible to plot all the return dynamics and we focus in Figure 2 on the sector ”Grains & Seeds”.

[Insert Figure 2 : Returns for the Sector ”Grains & Seeds”]

The evolutions can be very different in such a sector, which is clearly not homogenous. Even if we observe common volatility clustering, there is a switching trend in both mean and volatility for commodity ”soybeans” and partly for the commodity ”soybean oil” positively correlated with it.

5.4 Extreme portfolio allocation

Let us now consider four extreme portfolio allocations for the sector ”Grains & Seeds”, that are an equi-weighted portfolio, a minimum-variance portfolio, portfolios with equal risk contributions based on either the volatility, or the VaR at 5%, respectively. For each portfolio we provide the evolutions of the budget allocations, of the contributions to volatility and to VaR, respectively :

[Insert Figure 3 : Evolution of Budget Allocations]

[Insert Figure 4 : Evolution of Volatility Contributions]

[Insert Figure 5 : Evolution of VaR Contributions].

The main expected effect is to diminish the budget allocation in highly risky assets for all strategies controlling the risk (see Figure 3). At the extreme the commodity "soybeans" is not introduced in the min-variance allocation, whereas it appears small for strategies based on risk contributions, but clearly using the "substituability" with the less risky "soybean oil". We also observe the instability over time of the budget allocation for the min-variance portfolio, largely mentioned in the literature.

The risk parity portfolios have rather stable risk contributions for both risk measures [see Figures 4 and 5], especially when we compare their contributions to the VaR with the contribution of the equi-weighted and min-variance portfolios.

5.5 Portfolio management with weakened constraints on systematic risk contribution

Let us now consider the constrained optimization problem :

$$\begin{aligned} &\min_w VaR(w)^2 + \delta[(1-\pi)VaR_s(w) - \pi VaR_u(w)]^2 \\ &\text{s.t. } w'e = 1, w_i \geq 0, i = 1, \ldots, n, \end{aligned}$$

which corresponds to a mix between the minimization of the total VaR and the constraint on the risk contribution for systematic risk. The risk measure is the VaR at 5%, and the systematic component is driven by a single factor chosen equal to the DJUBS index return.

The optimal allocation depends on control parameters δ and π.

δ is a smoothing parameter : we get the min-VaR portfolio when $\delta = 0$, and strict restriction on the systematic risk contribution when $\delta \to \infty$.

The benchmark systematic risk contribution π takes values in (0,1). When the factor is a market index, π measures the selected degree of market neutrality of the portfolio. When $\pi = 0$, we are looking for a portfolio with no market influence on extreme risks.

We provide in Figures 6-8 the characteristics of these optimal portfolios, when the control parameters δ and π vary.

[Insert Figure 6 : Portfolio Allocations]

In Figure 6, the allocations are provided as function of δ and π. They are computed for the last available date : September, 24, 2012, and are based on the 252 days preceding the computation date. As expected the allocations of commodities with a large beta diminish, when we get closer to market neutrality. The surfaces feature some convexity property, which means that we have no found separation Theorem.

The last figures 7 and 8 are providing the risk contributions first between systematic and unsystematic risk, then between the commodities.

[Insert Figure 7 : Total Contribution to Systematic Risk]]

[Insert Figure 8 : Relatives Contribution to Systematic Risk]

For large δ, the total contribution to systematic risk is close to the benchmark contribution π. When $\delta = 0$, we get a small contribution to systematic risk as expected in the min-VaR portfolio.

Finally, we provide in Table 5 the two entries table of contributions to total risk.

6 Concluding remarks

We have introduced in this paper a unified optimization framework for asset allocation, which provides a mix between risk minimization and weakened budget and risk contribution restrictions. These allocation techniques includes the most well-known allocation procedures, such as the mean-variance and the minimum variance allocation as well as the equally weighted and risk parity portfolios.

There exist at least four reasons for considering such a mix focusing on the sytematic component of the risk.

The first one is to account for transaction costs, when looking for the portfolio adjustment. In this respect the introduction of constraints on the risk contributions can have such an interpretation.

The second one is to account for the regulation for financial stability, that is, for the introduction of constraints on the budgets allocated to the different types of assets, according to their individual risk, but also to the capital required for systematic risk, which is based on the risk contribution. This justifies a restriction written on the systematic component of the portfolio.

The third one is the possibility to manage the degree of market neutrality of the portfolio.

Finally, the standard mean-variance approach applied to a large number of assets is very sensitive to small changes in the inputs, especially to the estimate of the volatility-covolatility matrix of asset returns. The introduction of budget and/or risk contributions on either asset classes, or types of risks (systematic vs unsystematic) will robustify such an approach.

However, if such a mix is needed, there is no general method to select an optimal mix, which might depend on the preference of the investor, but also on the liquidity features and on the potential regulation. In this framework, the best approach consists in considering different mix, to apply them empirically for portfolio allocation and compare the properties of the associated portfolios in terms of stability over time of budget allocations, risk contributions and performances.

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Table 3: Summary Statistics of Asset Returns

NAME	Beta	Ann.Ret	Ann.Vol	Min.Ret	Max.Ret	Skew	Kurt	MaxDD	VaR1%	VaR5%	VaR95%	VaR99%
brentcrudeoil	126.90	10.50	26.17	-30.46	12.22	-1.20	29.66	-73.92	-4.43	-2.43	2.50	4.19
heatingoil	96.46	5.67	19.79	-17.88	7.67	-0.51	14.58	-64.51	-3.35	-1.88	1.97	3.40
lightcrudeoil	93.24	4.67	18.86	-18.42	8.65	-0.78	18.81	-64.82	-3.41	-1.77	1.78	3.33
naturalgas	55.49	-7.47	19.07	-7.49	12.89	0.53	11.92	-92.03	-3.52	-2.01	1.77	3.91
corn	57.62	1.11	18.56	-11.44	12.55	0.31	15.15	-68.32	-3.65	-1.66	1.75	3.78
rice	20.57	-1.62	12.81	-4.69	3.58	-0.04	6.65	-54.00	-2.46	-1.25	1.31	2.52
soybeanoil	53.87	1.86	15.62	-5.45	6.06	0.13	7.14	-54.04	-2.73	-1.45	1.61	2.79
soybeans	103.65	12.92	34.80	-14.98	14.34	0.04	6.70	-73.75	-6.02	-3.35	3.42	6.15
wheat	43.69	-0.84	15.66	-7.95	7.34	0.17	12.59	-64.82	-3.12	-1.38	1.49	2.98
leanhogs	14.98	-2.17	12.48	-5.14	4.98	-0.05	5.95	-59.34	-2.20	-1.31	1.25	2.11
livecattle	11.97	0.67	8.17	-5.67	2.57	-0.36	7.96	-30.31	-1.36	-0.83	0.85	1.31
copper	90.92	10.30	27.48	-12.58	11.54	-0.04	6.26	-64.31	-4.81	-2.62	2.77	4.65
gold	38.81	5.30	13.57	-6.06	8.53	0.03	12.23	-40.03	-2.68	-1.29	1.31	2.24
palladium	56.89	10.26	26.68	-10.69	13.32	0.00	8.95	-83.94	-5.03	-2.54	2.66	4.53
platinum	50.18	7.62	19.85	-8.47	9.39	-0.28	7.92	-63.67	-3.55	-1.94	1.87	3.22
silver	77.05	9.67	24.67	-16.50	12.51	-0.68	13.34	-56.34	-4.76	-2.30	2.31	4.24
cocoa	33.07	0.42	21.05	-8.09	7.97	-0.03	7.39	-60.72	-3.89	-2.05	2.09	3.72
coffee	39.63	1.41	21.71	-9.68	17.72	0.68	16.13	-65.29	-3.75	-2.03	2.14	3.82
cotton	85.34	5.20	37.37	-21.74	34.66	1.18	38.31	-95.66	-6.81	-2.75	2.94	6.95
orange_juice	23.69	1.42	29.67	-16.58	20.04	0.63	15.51	-82.78	-5.39	-2.74	2.63	5.60
sugar	67.78	14.42	37.95	-11.91	19.61	0.27	7.97	-65.71	-7.00	-3.60	3.93	6.66
DJUBS Index	100.00	0.01	15.62	-8.76	5.81	-0.25	7.25	-57.13	-2.74	-1.53	1.55	2.61

Table 5 : Decomposition of Total Risk

Assets	**Beta**	**ERC-VaR** Weight	**Syst. Factor**	**Idio. Error**	**Total**
Corn	1.09	14.9%	19%	1%	20%
Rice	0.35	28.19%	16%	4%	20%
SoybeanOil	0.76	22.98%	18%	2%	20%
Soybeans	0.87	18.14%	18%	2%	20%
Wheat	1.12	15.8%	19%	1%	20%
Total		100%	90%	10%	100%

Table 4 : Historical Correlation Matrix

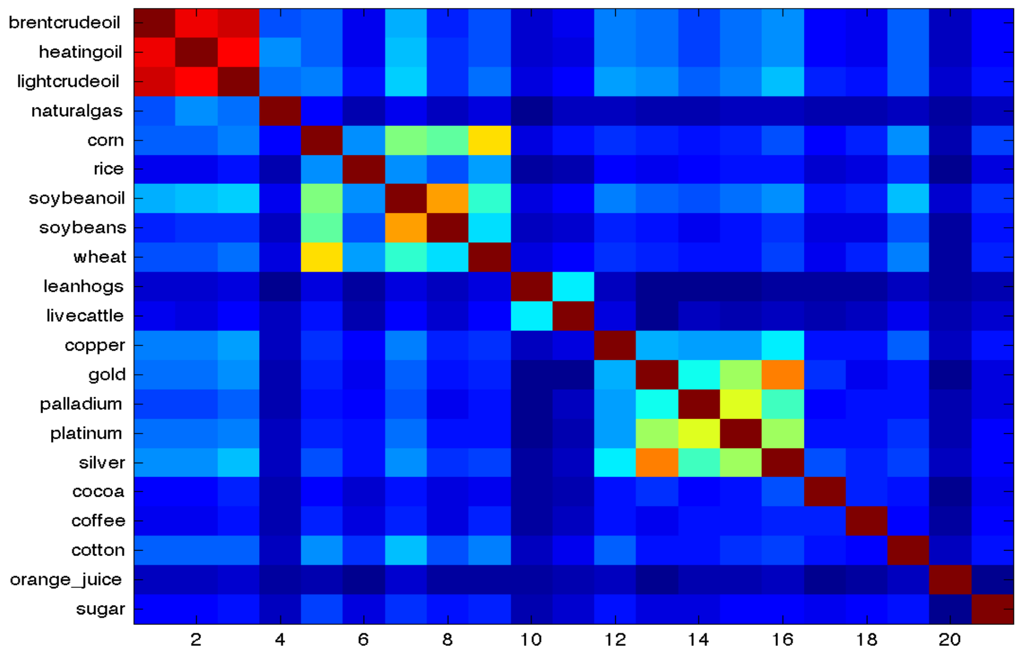

Figure 1 : Scatterplots of One-and Bi-dimensional Distributions; Sector : Grains & Seeds

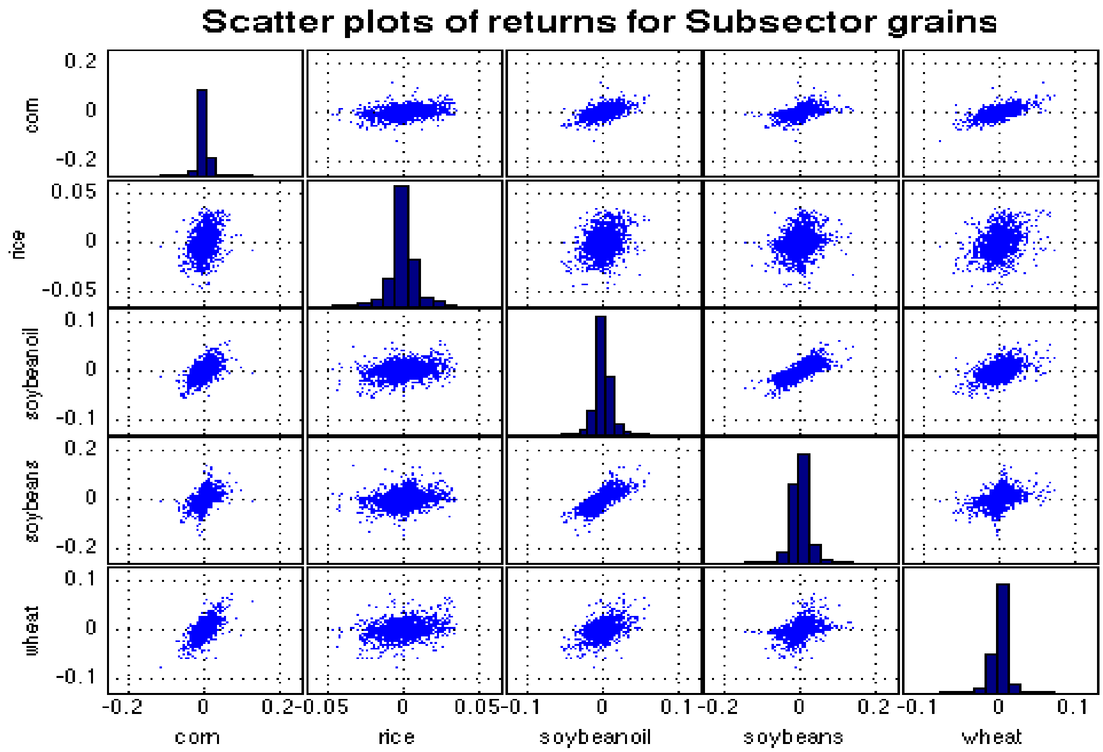

Figure 2 : Returns for the Sector ”Grains & Seeds”

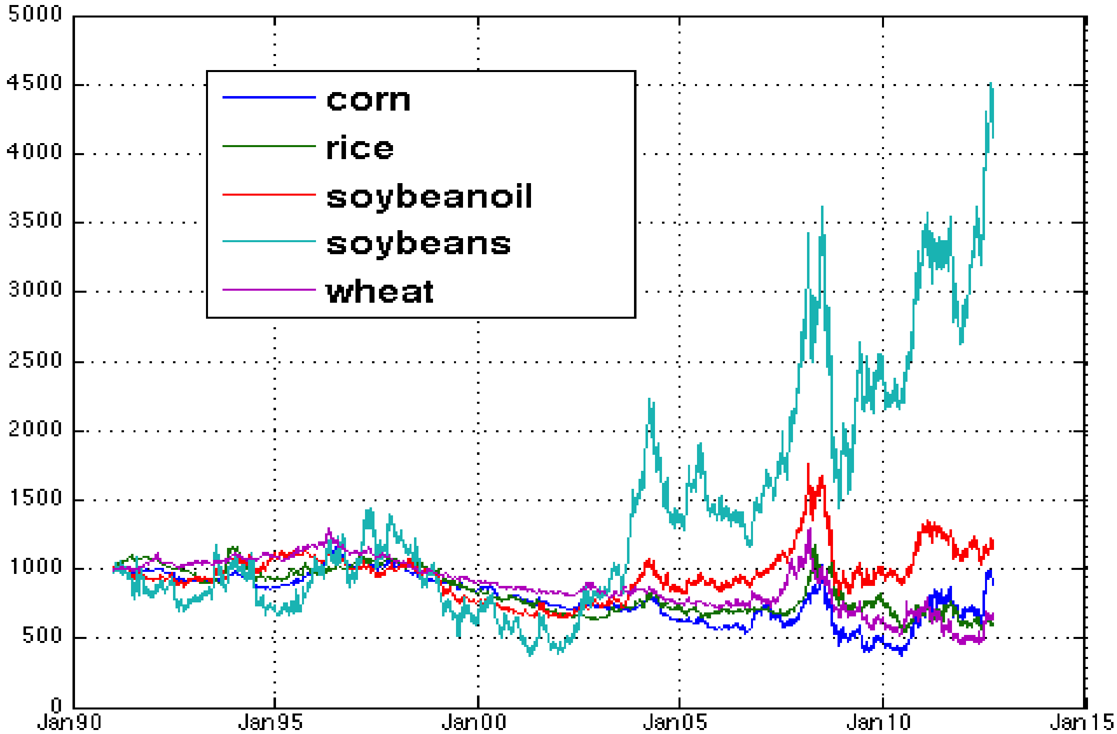

Figure 3 : Evolution of Budget Allocations

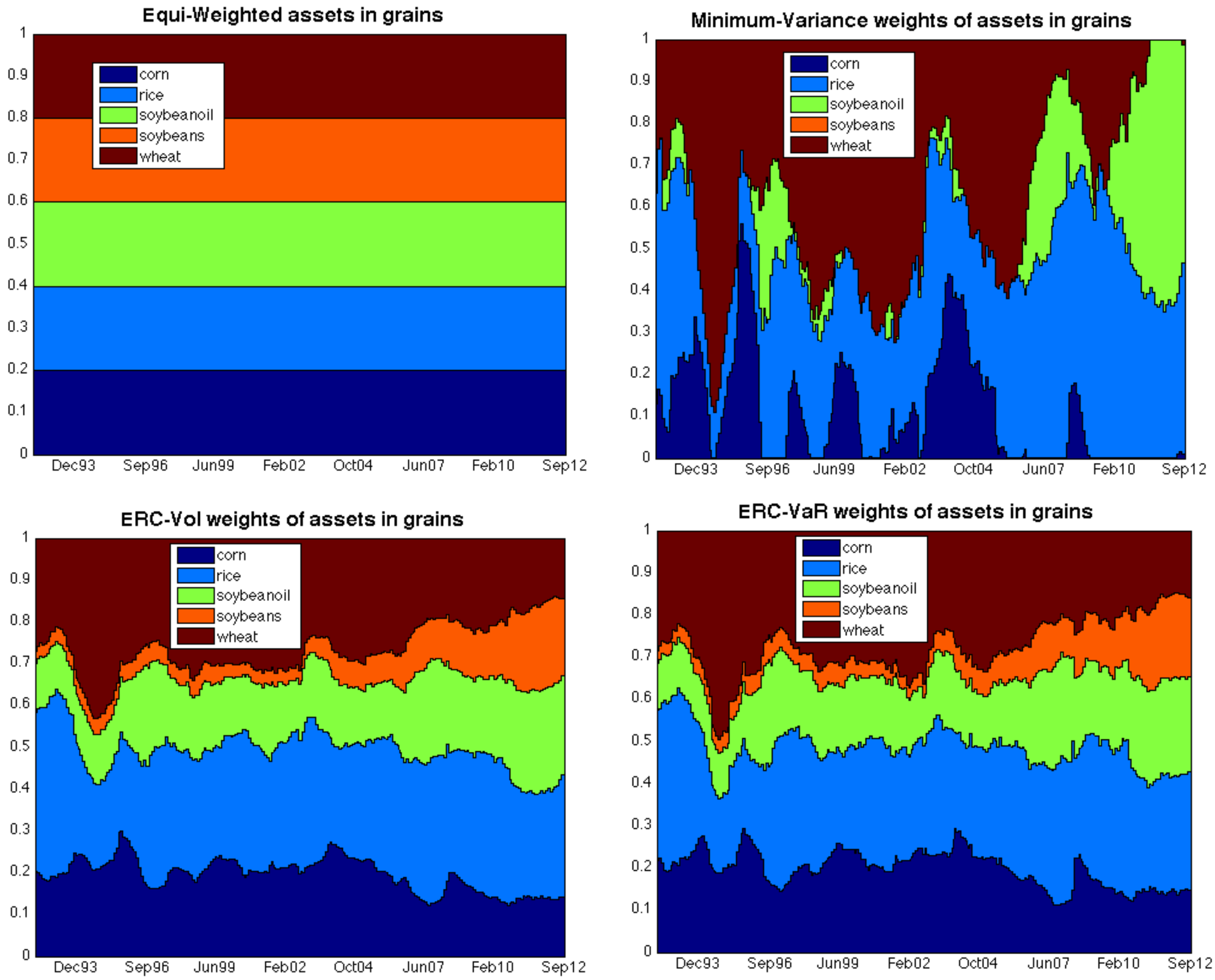

Figure 4 : Evolution of Volatility Contributions

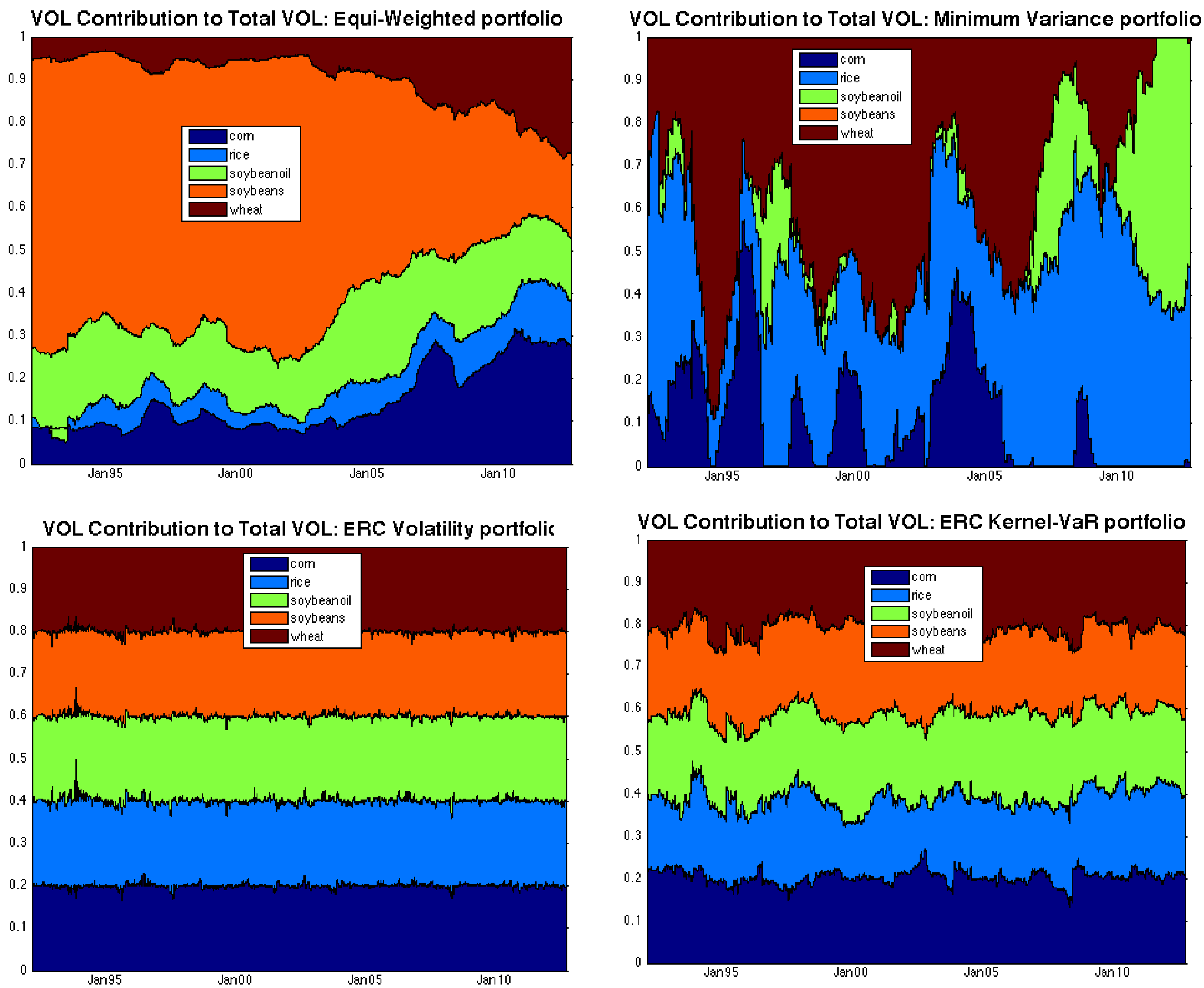

Figure 5 : Evolution of VaR Contributions

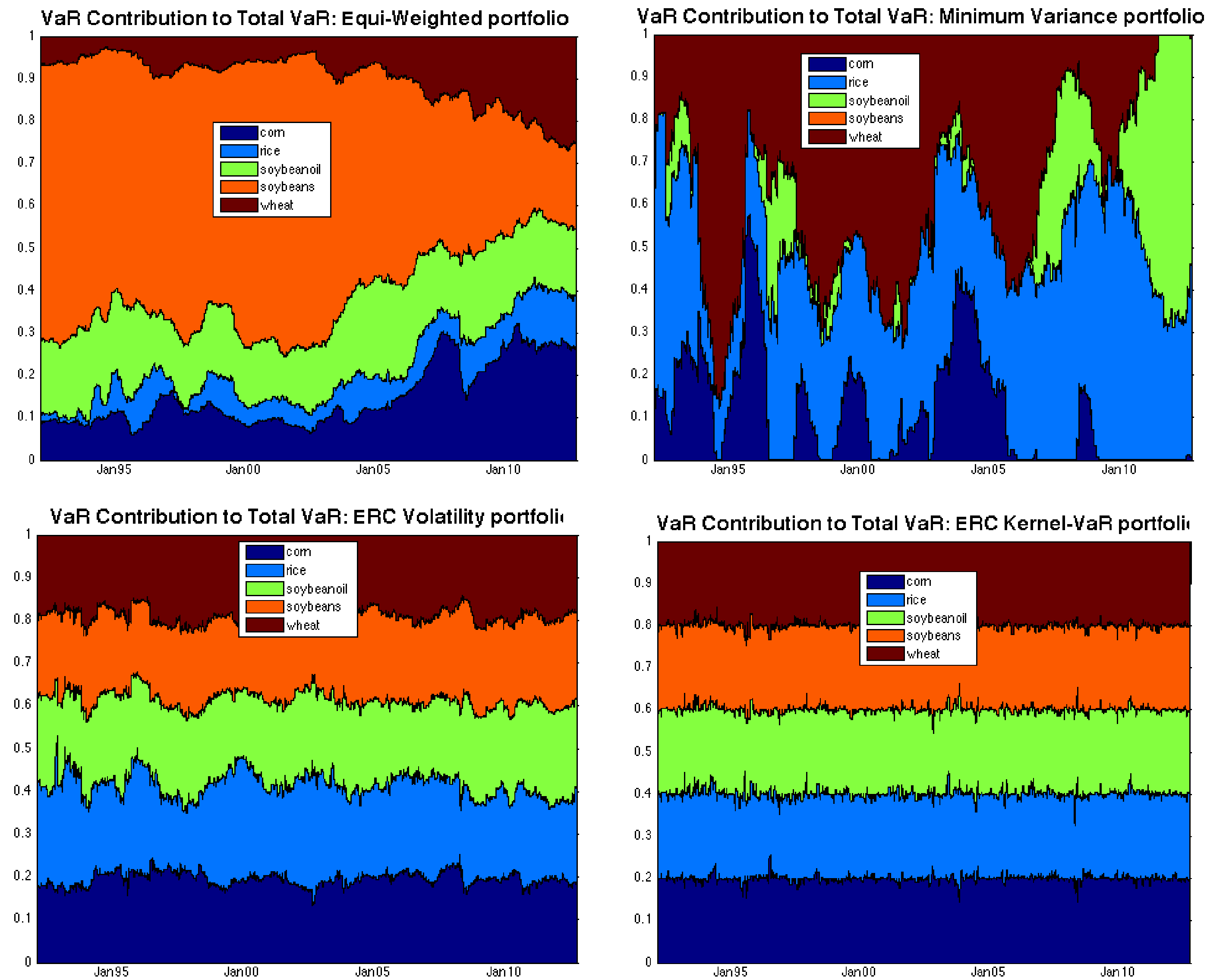

Figure 6 : Portfolio Allocations

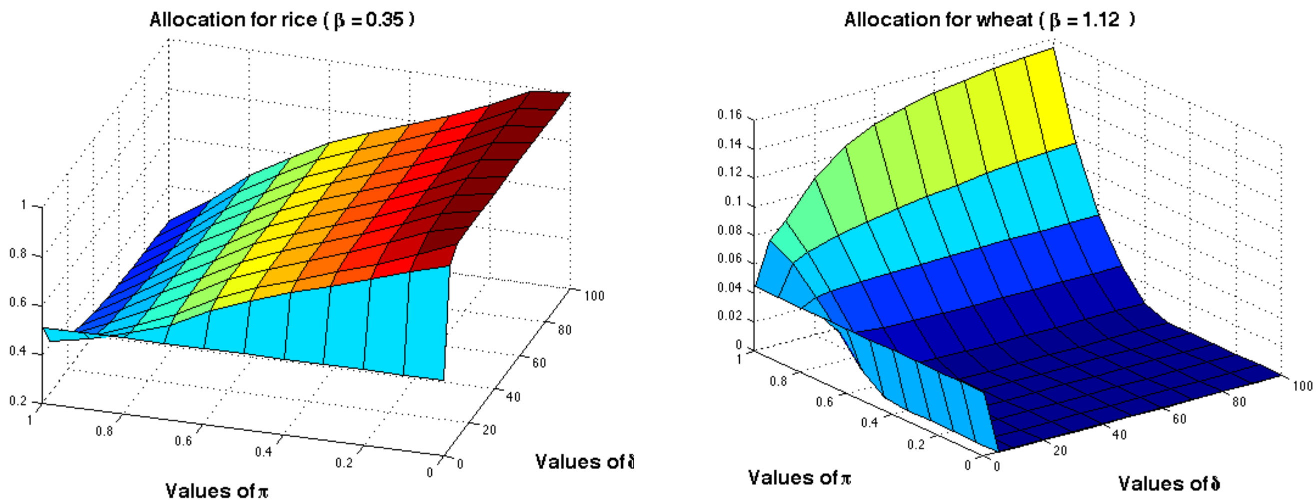

Figure 7 : Total Contribution to Systematic Risk

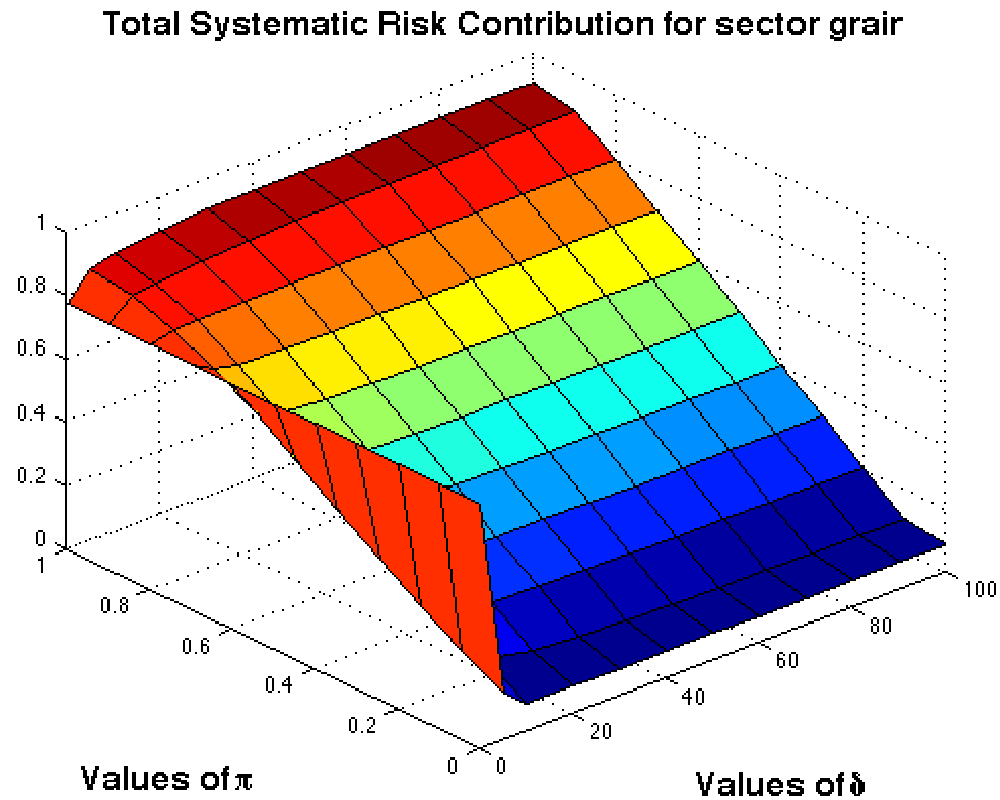

Figure 8 : Relative Contributions to Systematic Risk