Generalized Risk-Based Investing*

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Abstract

Risk-based portfolio strategies - such as Minimum Variance, Maximum Diversification, Equal-Weighted and Risk Parity, to name the most famous - have become increasingly popular in the investment industry due to their return-agnostic and risk management features. In this article, we show that these portfolio construction methodologies are special cases of a generic function defined by two specific parameters: the first one controls the intensity of regularization and the second one determines the tolerance for individual total risk. We investigate the theoretical properties of this class of strategies, giving expressions for solutions under general and specific risk models. This allows us to discuss important distinctive features of these portfolios, such as market beta, volatility, or exposure to low-vol/low-beta factors, while not being dependent on a specific sample. We illustrate these theoretical results by an empirical investigation of a large sample of international developed market equities over the 2002-2012 period.

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1 Introduction

Modern Portfolio Theory is the intellectual foundation of mainstream allocation and portfolio construction methodologies. Yet, heuristic (i.e. ad-hoc) approaches to portfolio construction seem still widely applied in finance. In practice, applying Markowitz mean-variance solution to real-life problems proves challenging, mostly because it requires the estimation of expected returns which are notoriously unstable and hard to predict (Merton, 1980). The asset management industry has thus increasingly relied on alternatives to the mean-variance efficient portfolio, with an emphasis on solutions that do not need returns' forecasts as inputs, that is pure risk-based portfolio construction techniques.

Remaining within the mean-variance framework, the Minimum Variance (MV) portfolio, suggested in Haugen and Baker (1991), is the sole efficient allocation that dispenses with precise and complete return forecasts. Since the largest source of uncertainty and error comes from estimating returns, MV has proven to be pretty robust in practice (see, among others, Clarke, De Silva, and Thorley, 2006)[1]. However, MV implicitly takes the strong assumption that all expected returns are equal. Using the milder hypothesis that total risk is remunerated equally across assets (i.e., identical Sharpe ratios), Choueifaty and Coignard (2008) derive the "Maximum Diversification"portfolio (MD), that is the one which maximizes the ratio between undiversified and diversified volatility of the portfolio[2]. These portfolios still depend on the estimates of the risk and dependencies between assets, with the drawback that errors yielding extreme estimates will also mechanically lead to larger allocation to the badly estimated components, making the optimization an "estimation-error maximization"(Michaud, 1989). Abandoning the mean-variance framework and privileging robustness, DeMiguel, Garlappi, and Uppal (2009) show that the Equally Weighted (EW) or $1/n$ allocation, which is estimation-free, is also largely competitive in terms of performances. An intermediary approach between the extremes of MV and EW is introduced by Maillard, Roncalli, and Teiletche (2010) with the Equal Risk Contribution principle, better known as the Risk Parity (RP) portfolio (Qian, 2005, 2006), which does spread the (ex-ante) total risk equally among the portfolio components, defining a different approach to diversification than MD.

In parallel to these theoretical developments, contrasted market performances of the beginning of the century taught investors the pitfalls of having to forecast underlying assets returns.

[1] MV appears pretty mature in the asset management industry and competition is starting to be fierce. Scherer (2011) lists almost ten asset managers offering such solutions and the number is growing rapidly.

[2] This ratio is called "Diversification Ratio" by the authors. A closely-related portfolio has been defined by Martellini (2008) and Amenc, Goltz, Martellini, and Retkowski (2011). It does consist in the Maximum Sharpe ratio, but where expected returns are proportional to estimated downside semi-volatilities.

Accordingly, they started showing interest in pure risk-based allocations: in the multi-asset space for more robustness in their strategic decisions; in equities as alternatives to market capitalization benchmarks[3], which are heavily concentrated in a few stocks and which are significantly biased towards overvalued stocks and sectors[4]. Significant money has thus been (and more is expected to be) allocated to these types of investment, with many asset management companies developing their own proprietary solutions.

Accompanying this trend, a growing set of academic research is becoming available, in which two branches of research can be primarily distinguished[5]: the first one is dealing with the issue of the historical performance of risk-based strategies, comparing the returns of the various portfolios; the second one is the analytic exploration of the properties of risk-based strategies, focusing on explaining how the weights are derived from the characteristics of the assets. In the first category, Choueifaty, Froidure, and Reynier (2013) conclude that MD dominates other approaches, quite consistently with the result of Linzmeier (2011) according to whom MD and MV portfolios achieve the best Sharpe ratios. At the opposite, Chow, Hsu, Kalesnik, and Little (2011) emphasize that the "non-optimized" allocations (EW, RP) are not dominated by the "optimized" ones (MV and MD), a result broadly confirmed by Leote, Xu, and Moulin (2012), Chaves, Hsu, Li, and Shakernia (2011), or Leclerc, L'Her, Mouakhar, and Savaria (2013). All in all, with the exception of the fact that risk-based approaches seem to consistently dominate market-capitalization portfolios, empirical results can be deemed contradictory and seem contingent on the universe and the period of study, which is inherent to empirical backtests. In that perspective, by being analytical, the second strand of the literature potentially offers a more general understanding of the individual characteristics of popular risk-based portfolios and of their differences. Maillard, Roncalli, and Teiletche (2010), Choueifaty, Froidure, and Reynier (2013), Lee (2011), Kaya and Lee (2012), Scherer (2011) and Clarke, De Silva, and Thorley (2011, 2013) provide exact or approximate analytical solutions for specific cases, allowing them to link allocation weights to individual assets characteristics and compute the sensitivity of the portfolios to various parameters. Our article belongs to this second branch of research, but offers new insights as we provide a general unifying framework encompassing all the popular risk-based allocation methodologies, but also a wider set of similar strategies.

Firstly, we show that all risk-based approaches can be mapped on a plane defined with only

[3]In this perspective, it is interesting to notice that traditional index providers such as MSCI, FTSE or S&P have entered into the arena of these alternatively-weighted indices (also called "Smart Beta").

[4]As emphasized, for instance, by the high exposure to technology-related stocks during the dotcom bubble.

[5]We restrict here to academic literature devoted to single-asset portfolio construction such as equities portfolios. There is a fairly large literature analyzing multi-asset risk-based portfolio construction but it tends to be restricted to some strategies, and mostly risk parity; see, notably, Asness, Frazzini, and Pedersen (2012), or Anderson, Bianchi, and Goldberg (2012).

two parameters. The first parameter is a regularization parameter which implies differences in sensitivity to variance-covariance estimates. The MV, which is highly sensitive to their values, and the EW, which is totally independent from them, represent both extremes of the spectrum. RP occupies a middle-ground between these two: it is sensitive to risk parameters but less so than the MV. The second parameter gives the tolerance for individual total risks. The MV, which is the most averse to risk, and what we name below the Most Concentrated portfolio (MC), which is only invested in the riskiest asset, represent the extremes on this spectrum. MD lies in between, being more diversified than MC but less focused on individual total risks than MV. Implicitly, the correlation is the definitive input for MD. We also show that, with the exception of EW, all the popular risk-based methodologies will have a preference for low-beta and low-volatility assets. In general, the market beta of these portfolios will be low and their total risks inferior to the market's ones, but this will be more prominent for the MV which is the most defensive. Being biased towards low-beta and low-volatility, risk-based solutions will show significant tracking-error to market portfolios in most circumstances, and particularly again for MV and MD. Secondly, we use our unifying framework to demonstrate that all risk-based portfolios can be obtained as specific solutions of the same general constrained minimum variance optimization program, extending the class of regularized MV portfolios considered by DeMiguel, Garlappi, Nogales, and Uppal (2009) and Fan, Zhang, and Yu (2012)[6]. We also provide closed-form solutions when available and a derivation of the conditions of existence and unicity of the different portfolios. Furthermore, the theoretical results yield as a by-product different algorithms to efficiently solve for optimal portfolio composition, expanding the recent work of Chaves, Hsu, Li, and Shakernia (2012). Thirdly, we illustrate the theoretical results with a large sample of international developed markets equities (based on MSCI World composition) over the period 2002-2012. Notably, we present empirical characteristics of the most popular risk-based portfolios and contrast them with our theoretical results.

The rest of the article is organized as follows. After introducing some notations, section 2 introduces the general formula for risk-based allocations and expresses them as a general constrained variance-minimization problem. We also investigate mean-variance optimality conditions for the different risk-based portfolios. Section 3 provides analytical results on the solutions to the global risk-based program, first under general conditions and then for specific risk models (unique correlation, factor models). In section 4, we run an empirical analysis on a large universe of international developed markets equities, the idea being not to establish a back-test per se but rather to provide illustrations of our theoretical results. Section 5 gives

[6]While the Lp regularization approach of DeMiguel, Garlappi, Nogales, and Uppal (2009) nests as special cases the MV and EW portfolios, it cannot span the entire space of risk-based portfolios due to the restrictive set of portfolio constraints considered and the absence of a risk tolerance parameter.

a summary of our different results and concludes the paper. Additional proofs and results are provided in the appendix.

2 General Specification of Risk-Based Portfolios

After recalling the definitions of the marginal and total risk contributions, we provide in this section a general characterization of risk-based allocation strategies. We also investigate the existence and uniqueness of the risk-based solutions. Finally, we analyze the conditions for mean-variance optimality of general risk-based portfolios.

2.1 Definitions and Notations

Consider a portfolio $\mathbf{w} = (w_1, ..., w_n)^{\mathbf{T}}$ of n assets. Let σ_i^2 be the variance of asset i, σ_{ij} denote the covariance between assets i and j, and $\mathbf{\Omega}$ is the positive definite variance-covariance matrix. Define $\sigma_p = \sqrt{\mathbf{w^T \Omega\, w}} = \sqrt{\sum_i \sum_j w_i w_j \sigma_{ij}}$ as the standard deviation (volatility) of the portfolio. The marginal risk contribution of the i-th asset defines the sensitivity of the portfolio total risk to a (supposedly small) change in the weight of asset i. It is given by

$$MRC_i = \frac{\partial \sigma_p}{\partial w_i} = \frac{\sigma_{ip}}{\sigma_p},$$

where $\sigma_{ip} = \sum_j w_j \sigma_{ij}$ denotes the covariance of the i-th asset with the portfolio. If we define the total risk contribution of the i-th asset as

$$TRC_i = w_i \times MRC_i,$$

we get the following risk decomposition[7]

$$\sigma_p = \sum_{i=1}^{n} TRC_i.$$

The portfolio risk can then be viewed either as a simple weighted average of the individual marginal risk contributions, or as the sum of all total risk contributions. In the following, we frequently use the beta of the i-th asset to the portfolio, β_{ip}, with $\beta_{ip} = \sigma_{ip}/\sigma_p^2 = MRC_i/\sigma_p$. We can similarly define the correlation coefficient of the i-th asset with the portfolio, ρ_{ip}, where $\rho_{ip} = \beta_{ip} \times \sigma_p/\sigma_i = MRC_i/\sigma_i$. We finally define b_i as the percentage risk budget of asset i, with $b_i = TRC_i/\sigma_p$ and $\sum_{i=1}^{n} b_i = 1$.

[7] Apart from volatility, risk decomposition, based on Euler decomposition principle, is also verified by the VaR (Gouriéroux, Laurent, and Scaillet, 2000), the Expected Shortfall (Scaillet, 2004) and, more generally, by any linear homogeneous risk measure (Tasche, 2004).

2.2 Mathematical Program

We define the general risk-based allocation program as

$$\begin{aligned} \mathbf{w}^* = & \arg\min_{\mathbf{w}} D\left(f\left(w_i;\gamma,\delta\right)\right), \\ s.t. & \textstyle\sum_{i=1}^{n} w_i = 1, \end{aligned} \tag{1}$$

with

$$f\left(w_i;\gamma,\delta\right) = \frac{w_i^{\gamma}}{\sigma_i^{\delta}} \times MRC_i,$$

where $\gamma \geq 0$ and $\delta \geq 0$ are key parameters we interpret later. $D\left(.\right)$ is a dispersion metric, such as the standard-deviation or the mean absolute deviation. In what follows, we do not need to make a specific choice for $D\left(.\right)$, as our results are independent of a particular specification. We call the functions $f\left(w_i;\gamma,\delta\right)$ "modified risk contributions ". Notice that we only impose a unit budgetary constraint but no long-only restrictions. This implies that, for all these portfolios, all the assets will be held in the portfolio, but some might be shorted. In the rest of the paper, we extend our results in a long-only format when appropriate and tractable.

The program (1) is encompassing all well-known risk-based allocation methodologies, and notably the Minimum Variance (MV), the Most Diversified (MD), the Risk Parity (RP) and the Equal-Weighted (EW) portfolios. In Table 1, we detail how these specific cases can be obtained from program (1) and gives a reminder of their main theoretical properties.

Independently from the dispersion measure, it is straightforward to see that solving (1) is equivalent to

$$\frac{w_i^{\gamma}}{\sigma_i^{\delta}} \times MRC_i = \frac{w_j^{\gamma}}{\sigma_j^{\delta}} \times MRC_j = \tau \qquad \forall\left(i,j\right) = \left(1,2,...,n\right), \tag{2}$$

with

$$\sum_{k=1}^{n} w_k = 1,$$

where τ is a positive target constant, which we can define as the ratio between the risk of the portfolio and the average weighted relative risk-contribution (see Appendix A for more details)[8]. Knowing the exact value of τ is unnecessary however to solve for portfolio's weights w_i, since the budgetary constraint, $\sum_k w_k = 1$, is only acting as a normalization constraint. In particular, if the portfolio $\widetilde{\mathbf{w}}$ is such that $\widetilde{w}_i^{\gamma}\,\sigma_i^{-\delta}\,MRC_i = \tau$ with $\sum_k \widetilde{w}_k \neq 1$, the portfolio defined by $\widetilde{\mathbf{w}}/\left(\mathbf{1}^{\mathbf{T}}\,\widetilde{\mathbf{w}}\right)$ is still respecting condition (2) but for another value of τ. This globally

[8]Even if we want to restrict to risk-based strategies, the framework proposed here is flexible enough to encompass other characteristic portfolios such as the Max Sharpe Ratio portfolio (Martellini, 2008) or the Risk Budgeting portfolio (Bruder and Roncalli, 2012). To see this, we only need to replace in (2) the individual

Table 1: Risk-Based Allocation Schemes Characteristics

Portfolio	(γ,δ)	Strategy definition	Properties	Risk budget b_i
MV	$(0,0)$	$MRC_i = MRC_j$	$\beta_{ip}=1$	w_i
MD	$(0,1)$	$\sigma_i^{-1}MRC_i = \sigma_j^{-1}MRC_j$ $(\rho_{ip}=\rho_{jp})$	$\beta_{ip}=\frac{\sigma_i}{\sum_{k=1}^n w_k\,\sigma_k}$	$\frac{w_i\sigma_i}{\sum_{k=1}^n w_k\sigma_k}$
RP	$(1, 0)$	$w_i MRC_i = w_j\, MRC_j$	$\beta_{ip}=\frac{1}{w_i}\times\frac{1}{n}$	$\frac{1}{n}$
EW	$(\infty,0)$	$w_i = w_j = n^{-1}$	$\beta_{ip}=\frac{1}{w_i}\times\frac{\sum_{k=1}^n\sigma_{ik}}{\sum_{i=1}^n\sum_{k=1}^n\sigma_{ik}}$	$\frac{\sum_{k=1}^n\sigma_{ik}}{\sum_{i=1}^n\sum_{k=1}^n\sigma_{ik}}$

means that a risk-based solution primarily seeks risk repartition, while the absolute level of risk is only a matter of budget constraint and potential use of leverage[9]. All in all, the risk-based allocation methodologies can be described through the equivalent of a two-fund theorem where the investor first determines the repartition of the risk and then set the global level of risk by choosing the appropriate budget/leverage.

2.3 Existence and Uniqueness

To analyze conditions of existence and uniqueness of the generalized risk-based portfolios, we start from the following alternative expression of the optimization problem

$$\begin{aligned} \widetilde{\mathbf{w}}^* = \arg\min_{\widetilde{\mathbf{w}}} \quad & \tfrac{1}{2}\,\widetilde{\mathbf{w}}^{\mathbf{T}}\,\mathbf{\Omega}\,\widetilde{\mathbf{w}}, \\ s.t.\ \sum_{i=1}^n & \left[\frac{\sigma_i^\delta\left(\widetilde{w}_i^{1-\gamma}-1\right)}{(1-\gamma)}\right] \geq c, \end{aligned} \tag{3}$$

where $\mathbf{\Omega}$ is the variance-covariance matrix and c is an arbitrary constant determined by the risk-based investment strategy, with weights rescaled to sum to one afterwards if necessary. Indeed, it is pretty straightforward to prove the equivalence of programs (2) and (3). If we

volatility σ_i by the appropriate individual characteristic ϱ_i. The program (2) then becomes

$$\frac{w_i^{\gamma}}{\varrho_i^{\delta}} \times MCR_i = \frac{w_j^{\gamma}}{\varrho_j^{\delta}} \times MCR_j \;\; \forall\,(i,j)\,,$$

with $\sum_k w_k = 1$. This program coincides with the Maximum Sharpe Ratio program if $\varrho_i = \mu_i$, $\gamma = 0$ and $\delta = 1$, where μ_i is the expected excess return of asset i. Following the same line of reasoning, the Risk Budgeting portfolio can be obtained by setting $\varrho_i = b_i$, $\gamma = 1$ and $\delta = 1$, where b_i denotes the percentage risk budget of asset i. All the different theoretical results given in this paper can then be translated to these portfolios by applying the appropriate identifying restrictions.

[9] Obviously, we assume here that there exists a risk-free rate at which the investor can lend and borrow at zero cost and without restriction.

write the Lagrangian function of the system (3), we get

$$L\left(\widetilde{\mathbf{w}};\lambda_c\right)=\frac{1}{2}\widetilde{\mathbf{w}}^{\mathbf{T}}\boldsymbol{\Omega}\widetilde{\mathbf{w}}-\lambda_c\left\{\sum_{i=1}^{n}\left[\frac{\sigma_i^{\delta}\,\left(\widetilde{w}_i^{1-\gamma}-1\right)}{(1-\gamma)}\right]-c\right\}, \tag{4}$$

where $\lambda_c \geq 0$ is the associated Lagrange multiplier. The first-order condition of (4) is given by

$$\nabla L\left(\widetilde{\mathbf{w}};\lambda_c\right)=\boldsymbol{\Omega}\widetilde{\mathbf{w}}-\lambda_c\,\nu \tag{5}$$

where $\nu=\left(\sigma_1^{\delta}/\widetilde{w}_1^{\gamma},...,\sigma_n^{\delta}/\widetilde{w}_n^{\gamma}\right)^{\mathbf{T}}$ is a column vector. Using the fact that $MCR_i=\frac{(\boldsymbol{\Omega}\widetilde{\mathbf{w}})_i}{\sigma_p}$, where $\left(\boldsymbol{\Omega}\widetilde{\mathbf{w}}\right)_i$ is the i-th element of vector $(\boldsymbol{\Omega}\widetilde{\mathbf{w}})$, equalizing (5) to 0 and rearranging leads to

$$\frac{(\widetilde{w}_i^{*})^{\gamma}}{\sigma_i^{\delta}}\times MCR_i^{*}=\frac{\lambda_c}{\sigma_p}. \tag{6}$$

This means that the optimal solution of (3) is characterized by an equality of the modified risk contributions, thus fulfilling condition (2). To guarantee the uniqueness of the solution, one needs to check for the convexity of the program. The second-order condition of (4) is given by

$$\nabla^2 L\left(\widetilde{\mathbf{w}};\lambda_c\right)=\boldsymbol{\Omega}+\gamma\,\lambda_c\,\omega, \tag{7}$$

with $\omega=\left(\sigma_1^{\delta}/\widetilde{w}_1^{\gamma+1},...,\sigma_n^{\delta}/\widetilde{w}_n^{\gamma+1}\right)^{\mathbf{T}}$. Convexity and existence of the solution is insured by positivity of the second-order derivative. From (7), it is straightforward to identify the conditions for it. Firstly, whatever the value of δ, when γ is either nil, odd or tends to infinity, the solution does exist as long as the covariance matrix $\boldsymbol{\Omega}$ is definite-positive. This notably implies that all characteristic portfolios (MV, MD, RP, EW) will be uniquely defined in such a case. Second, imposing a long-only constraint is also a sufficient condition for guaranteeing the existence of the solution when $\boldsymbol{\Omega}$ is positive semi-definite for all possible values of δ and γ (which are positive parameters).

Most popular risk-based methodologies can thus be retrieved as special cases of a constrained variance minimization problem with a particular portfolio constraint expressed through a specific equation. For instance, in the MV case, we have $c=(1-n)$ and the portfolio constraint is $\sum_k \widetilde{w}_k = 1$. In the case of MD, we have $c=(1-\sum_k \sigma_k)$ and the portfolio constraint is $\sum_k \widetilde{w}_k\ \sigma_k = 1$. Finally, for RP, we get $\sum_k \ln\left(\widetilde{w}_k\right) \geq c$, where c is a constant determined by the risk-based allocation[10]. More generally, in the case where $\delta=0$, we get

$$\begin{aligned}&\widetilde{\mathbf{w}}^{*}=\arg\min_{\widetilde{\mathbf{w}}}\quad \tfrac{1}{2}\,\widetilde{\mathbf{w}}^{\mathbf{T}}\,\boldsymbol{\Omega}\,\widetilde{\mathbf{w}},\\ &s.t. \textstyle\sum_{k=1}^{n}\ln(\widetilde{w}_k)\geq c.\end{aligned} \tag{8}$$

[10]We use here the following property: $\lim_{\gamma\to 1}\frac{\widetilde{w}_i^{1-\gamma}-1}{1-\gamma}=\ln\left(\widetilde{w}_i\right)$.

As discussed by Maillard, Roncalli, and Teiletche (2010), this program has two polar cases, which are MV for $c \to -\infty$, and EW for $c = -n\ln(n)$, while RP is between both[11]. The authors notably show that this feature naturally implies a hierarchy between the portfolios' volatilities, with $\sigma_{MV} \leq \sigma_{RP} \leq \sigma_{EW}$. We can use similar arguments in the case where $\gamma = 0$. In such a case, the program (3) can be rewritten as

$$\begin{aligned} \widetilde{\mathbf{w}}^* &= \arg\min_{\widetilde{\mathbf{w}}} \quad \tfrac{1}{2}\,\widetilde{\mathbf{w}}^{\mathbf{T}}\,\mathbf{\Omega}\,\widetilde{\mathbf{w}}, \\ s.t.\ & \textstyle\sum_{k=1}^{n} \widetilde{w}_k\ \sigma_k^{\delta} = c. \end{aligned} \tag{9}$$

From (9), we can infer that higher δ leads to significant weight increase for assets with higher individual total risks in order to fulfill the portfolio constraint. At the extreme, for $\delta \to \infty$, the optimal portfolio becomes the one with full concentration into the most risky individual asset. As previously mentioned, we quote this portfolio as MC for Most Concentrated portfolio. We also infer that $\sigma_{MV} \leq \sigma_{MD} \leq \sigma_{MC}$, thus getting another hierarchy of the volatilities of risk-based portfolios. Unfortunately, it is difficult to get more than these two hierarchies along γ and δ and, in particular, no general statement can be made with respect to the difference between the volatilities of MD and RP.

2.4 Optimality Conditions

We here investigate the conditions for optimality of the risk-based portfolios. When a risk-free rate is available, a risky portfolio is mean-variance efficient if it corresponds to the Maximum Sharpe Ratio portfolio, corresponding to the tangency portfolio. In practice, it is well-known that the solution is defined such that the ratio of the marginal expected excess return to the marginal risk contribution is the same for all the assets constituting the portfolio, that is

$$(MRC_i)^{-1} \times \mu_i = (MRC_j)^{-1} \times \mu_j, \qquad \text{for } i,\, j = (1, 2, ..., n)\,,$$

where μ_k corresponds to the expected excess return of asset k. As generalized risk-based portfolios are defined by $w_i^{\gamma}\ \sigma_i^{-\delta} MRC_i = w_j^{\gamma}\ \sigma_j^{-\delta} MRC_j$ for all i and j, the optimality condition becomes

$$w_i^{\gamma}\ \sigma_i^{(1-\delta)}\, SR_i = w_j^{\gamma}\ \sigma_j^{(1-\delta)}\, SR_j, \qquad \text{for } i,\, j = (1, 2, ..., n)\,, \tag{10}$$

with $SR_k = \mu_k/\sigma_k$ is the Sharpe ratio of asset k. Equality (10) shows that a risk-based strategy is optimal when assets have the same weighted risk-adjusted Sharpe ratio, e.g. $w_i^{\gamma}\sigma_i^{(1-\delta)} SR_i = \xi$,

[11] The authors use the acronym ERC (for Equal Risk Contributions) rather than RP (for Risk Parity) but the portfolio construction is the same and we prefer to stick to the latter expression which has become more popular.

for $(i = 1, 2, ..., n)$, with $\xi > 0$. Expression (10) remains fairly general and, in practice, plenty of combinations of return/risk characteristics can verify these equalities.

However, substituting for specific values of γ and δ in risk-based allocations leads to precise conditions of optimality. For instance, in the case of MV, the mean-variance optimality condition implies that

$$\sigma_i \times SR_i = \sigma_j \times SR_j \Leftrightarrow \mu_i = \mu_j, \qquad \text{for } i,\, j = (1, 2, ..., n)\,.$$

As a consequence, MV is optimal if the expected excess returns of all assets are identical. In the case of the MD strategy, the optimality implies that

$$SR_i = SR_j, \qquad \text{for } i,\, j = (1, 2, ..., n)\,. \tag{11}$$

In other words, it is sufficient that all assets have the same Sharpe ratios for MD to be mean-variance efficient.

The case of RP is a little bit more complex. The optimality condition is then

$$w_i\ \sigma_i\ SR_i = w_j\ \sigma_j\ SR_j, \qquad \text{for } i,\, j = (1, 2, ..., n)\,. \tag{12}$$

A common belief is that RP is mean-variance optimal if all assets have the same Sharpe ratio and the same correlation among them (Maillard, Roncalli, and Teiletche, 2010; Kaya and Lee, 2012), that is

$$SR_i = SR_j \quad \text{and} \quad \rho_{ij} = \rho, \ \text{for } i,\, j = (1, 2, ..., n)\,.$$

Indeed, since with constant correlation, the RP weights are given by $w_i = \sigma_i^{-1} / \sum_k \sigma_k^{-1}$ (Maillard, Roncalli, and Teiletche, 2010), it is then sufficient to have the same individual Sharpe ratio in order to verify restriction (12). However, expression (12) shows this a sufficient but not necessary condition and that other parameter combinations can lead to a mean-variance efficient RP portfolio.

EW is also concerned by this multiplicity of conditions leading to mean-variance optimality. In full generality, plenty of combinations of returns and volatilities can fulfill condition (10) when $\gamma \to \infty$. Still, a remarkable one does exist which is defined by the equality of both excess returns and volatilities for all assets plus a unique correlation (in which case, all assets have the same marginal risk contribution).

As a conclusion, we have shown in this section that all popular risk-based methodologies can be encompassed in a simple heuristic formula. We have also shown that all of them can be rewritten through a unique constrained minimum-variance optimization program. This general formulation opens the door to the definition of a wider set of risk-based strategies and offer a general theoretical framework to compare existing methodologies, as proposed in the next section.

3 Theoretical Properties of Risk-Based Portfolios

In this section, we derive the theoretical properties of risk-based portfolios. We first determine the general solution and give some numerical illustrations in a three-asset case. We notably review in details the role of the two key parameters γ and δ. We then analyze the portfolios for two specific variance-covariance matrix configurations: one where we assume unique correlation across assets, the other one where we assume one-factor representation.

3.1 The General Case

We begin by determining the general analytical solution before producing some numerical illustrations in a simple three-asset case. We end this subsection with an analysis of the general market beta of risk-based solutions.

3.1.1 Global Analytical Solution

Introducing the beta β_{ip} of asset i with the portfolio, $\beta_{ip} = MRC_i/\sigma_p$, it is straightforward to see that the optimal weights which satisfy the program (1) are given by

$$w_i^* = \frac{\left(\sigma_i^{-\delta}\,\beta_{ip}\right)^{-\frac{1}{\gamma}}}{\sum_{k=1}^{n}\left(\sigma_k^{-\delta}\,\beta_{kp}\right)^{-\frac{1}{\gamma}}}. \tag{13}$$

Expression (13) shows that the weights of a risk-based portfolio are inversely proportional to the individual risk-adjusted betas. Unfortunately, weights defined in (13) are not yielding a closed-form solution as the solution is endogenously defined. Indeed, the unconstrained optimal solution (13) depends on the portfolio weight distribution $\mathbf{w}$. However, from the general definition of a beta, which is merely the product of the correlation of asset i and the portfolio with the ratio between their respective individual total risks, i.e. $\beta_{ip} = \rho_{ip}\left(\frac{\sigma_i}{\sigma_p}\right)$, we can infer that any risk-based allocation will favor assets with low correlation to the rest of the universe and lower individual total risk. Another limitation of (13) is that it is not properly defined for the boundary case where $\gamma = 0$, but we give below another (closed-form) solution in that case.

The main interest of expression (13) is to illustrate the dependence of the solution on parameters γ and δ. Each of them has a different interpretation. γ can be interpreted as a regularization parameter. Varying γ will determine the sensitivity of the optimal weights to differences between individual volatilities and individual betas. For instance, as $\gamma \to \infty$, the portfolio weights approach EW, and the asset allocation becomes insensitive to the volatility and correlation estimates. At the opposite, when $\gamma \to 0$, the portfolio holdings approach the MV (for $\delta = 0$) or the MD (for $\delta = 1$), and the optimal solution becomes highly sensitive to the variance-covariance matrix estimates. As usual, the advantage of the regularization obtained through increasing γ is that it provides more robustness to measurement errors. Another positive implication is that, by being less sensitive to risk inputs, it leads to more stable solutions with much lower turnover, consistently with the empirical results of Chow, Hsu, Kalesnik, and Little (2011), Leote, Xu, and Moulin (2012) and Lohre, Neugebauer, and Zimmer (2012) on the superiority of EW and RP over MV and MD in that regard.

The negative implication of the regularization is twofold. First, by giving less and less confidence to risk parameters as γ increases, we are less and less able to identify common risk characteristics of individual assets. This paves the way to "duplication risk" as shown by Choueifaty, Froidure, and Reynier (2013) for popular risk-based portfolios: if we add an asset totally similar to one already existing in the portfolio then, contrary to MV and MD cases, EW and RP allocations will be modified while common sense suggests they should not. We can generalize this argument by saying that any risk-based allocation methodology characterized by γ being different from zero will not satisfy the duplication invariance property. The higher γ, the more the duplication risk will be present. For instance, EW will suffer more from this bias than the RP (as illustrated by Choueifaty, Froidure, and Reynier (2013)), which is due to the fact that the former has no views on the risk, and in particular that it will ignore the fact that individual total risks are similar and correlation is high, while the latter is incorporating some of the information coming from the variance-covariance matrix[12].

Second, the endogenous nature of the solution is coming from γ [13]. Indeed, we can show that when $\gamma = 0$, as e.g. for MV and MD, we have the following analytical expression for the

[12] The lack of duplication invariance for RP is getting increasing interest in the literature. To limit that risk, the natural approach is to pre-filter assets for their common properties through factor modeling (such as Principal Component Analysis). See for instance Meucci (2009), Lohre, Opfer, and Orszag (2011), Darolles, Gourieroux, and Jay (2012), or Roncalli and Weisang (2012).

[13] The sole case where optimal weights with γ different from 0 are not endogenous is when γ tends to ∞, that is EW, which is a limit case.

optimal portfolios (see Appendix A):

$$\mathbf{w} = \frac{\mathbf{\Omega}^{-1}\sigma^{\delta}}{\mathbf{1^T}\mathbf{\Omega}^{-1}\sigma^{\delta}}, \tag{14}$$

where $\sigma^{\delta} = \left(\sigma_1^{\delta}, ..., \sigma_n^{\delta}\right)^{\mathbf{T}}$ is a column vector. This means that weights are directly determined by the product of the inverse of the variance-covariance matrix with the vector of individual volatilities scaled by the δ parameter. In practice, δ can be interpreted as a parameter describing the tolerance for individual total risks σ_i. The higher δ, the less penalized will be large individual total risks σ_i. For instance, it implies that MV will have less tolerance for high volatility assets than MD, everything else equal. At the extreme, for $\delta \to \infty$, the optimal portfolio becomes the one with full concentration into the most risky individual asset, that is the Most Concentrated portfolio (MC).

3.1.2 Numerical Example

We here illustrate the behavior of the generalized risk-based function with respect to its parameters in the general case through a simple numerical example. For that purpose, we use the same set-up as Alvarez, Luo, Cahan, Jussa, and Chen (2011) made of three assets, such as the first two have equal volatility and are positively correlated and the third one is uncorrelated with higher volatility. The correlation matrix is $\left(\begin{smallmatrix} 1 & 0.5 & 0 \\ 0.5 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ and the volatility vector is $(0.1, 0.1, 0.2)^{\mathbf{T}}$. This set of parameter notably implies that assets 1 and 2 are strongly similar, even if not perfectly identical. To get portfolio compositions, we use the Newton-Raphson algorithm derived in Appendix D. We represent the portfolio's volatility and two different dispersion measures: the diversification ratio[14] and the breakdown of the total risk contributions TRC_i.

Figure 1 shows the characteristics of the optimal portfolios defined on this universe when varying either γ or δ. On the left-hand side, we let γ vary from 0 to 6 while δ is fixed at 0. We thus explore the axis of possible solutions spanning from the MV ($\gamma = 0$) to the EW ($\gamma \to \infty$) portfolios. We verify that the weights converge to $1/n$ for higher values of γ, that $\gamma = 0$ yields the lowest variance, and that the risk contributions of the three assets are exactly equal for $\gamma = 1$ (RP case). As the weights converge to $1/n$, the risk contribution of the high volatility asset increases, as does the volatility of the portfolio. Unlike the other characteristics of interest, the diversification ratio does not evolve in a monotonous way, but reaches a maximum for γ between 1 and 2. On the right-hand side of Figure 1, we let δ vary from 0 to 6 while γ is kept constant at 0, thus exploring the set of solutions encompassing the MV and the MC portfolios,

[14]The diversification ratio introduced in Choueifaty and Coignard (2008) is computed as the ratio between diversified and undiversified risks. More precisely it is defined as: $DR_p = \left(\sum_{k=1}^{n} w_k\sigma_k\right)/\sigma_p$.

passing through the MD portfolio. As δ increases, the weight of the high volatility asset (here, asset 3) increases, as does its risk contribution and the portfolio volatility. We observe that portfolios are attaining in such a case much higher level of volatility than when γ varies. The diversification ratio increases up to a maximum at $\delta = 1$, which is the MD portfolio, then decreases as the discounting of the individual volatilities leads to an over-representation of the high volatility asset, which will eventually constitute the whole portfolio by itself.

We next check the sensitivity of the solutions to changes in the variance-covariance matrix. The first change is to let the volatility of the third asset vary from 5% to 30%, keeping the rest of the covariance matrix unchanged. To test for the sensitivity of the solution to correlation, we separately let the correlation between the first two assets vary from -0.5 to 0.9, volatilities being kept as in the reference example. Notice that both scenarios imply that assets 1 and 2 are still similar, both in weights or as total risk contributions. Results are summarized in Figure 2 where, on the left-hand side, we let γ vary while maintaining δ fixed at 0, and on the right-hand side, we let δ vary while maintaining γ fixed at 0. Starting with varying γ first (left-column), we see that a high γ decreases the sensitivity of the weights to changes in volatility and correlation. Indeed, higher values of γ are characterized by more stability in the portfolio weights whatever volatility and correlation values. This naturally implies some more significant changes for total risk contributions (or risk budgets). Notice that the contribution of the third asset to total risk is, by definition, constant for $\gamma = 1$ (RP), decreases with its volatility for $\gamma < 1$, and increases for $\gamma > 1$. Likewise, the sign of the elasticity of the weights of each asset to a change in the correlation between the first two will depend on γ. The relative risk contributions of assets one and two increase with their correlation for $\gamma > 1$, is constant for $\gamma = 1$ and decreasing for $\gamma < 1$. Since all contributions sum to one, an increase of the dependency between the two correlated assets will lead to more risk allocated to the uncorrelated asset when γ is low, and less when γ is high, showing the lower sensitivity in the latter case.

On the right-column, we illustrate the sensitivity of the solution to variance and correlation changes for different levels of δ. We first observe that the range of variations is much larger. As already noticed for the base case example (see Figure 1), changes in δ are triggering more significant changes in allocations. The weight of the uncorrelated asset decreases with its volatility when $\delta < 1$, is constant for $\delta = 1$ (MD), and increases when $\delta > 1$. Unlike γ, whatever the value of δ, a positive correlation shock never leads to an increase of the risk allocated to the correlated assets, but always favor the uncorrelated asset (asset 3 in this example).

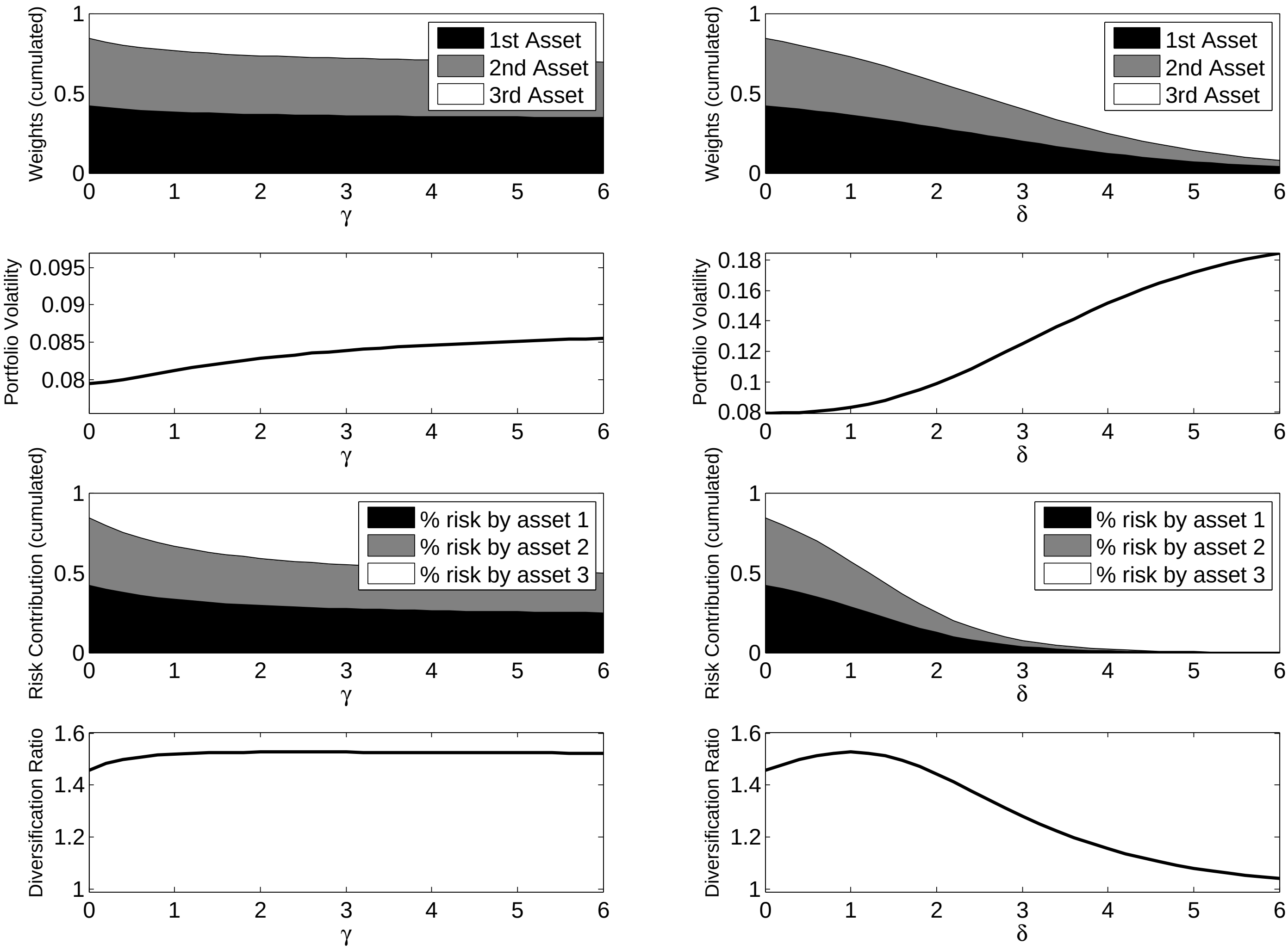

Figure 1: Characteristics of the Optimal Risk-based Portfolios when Varying γ and δ

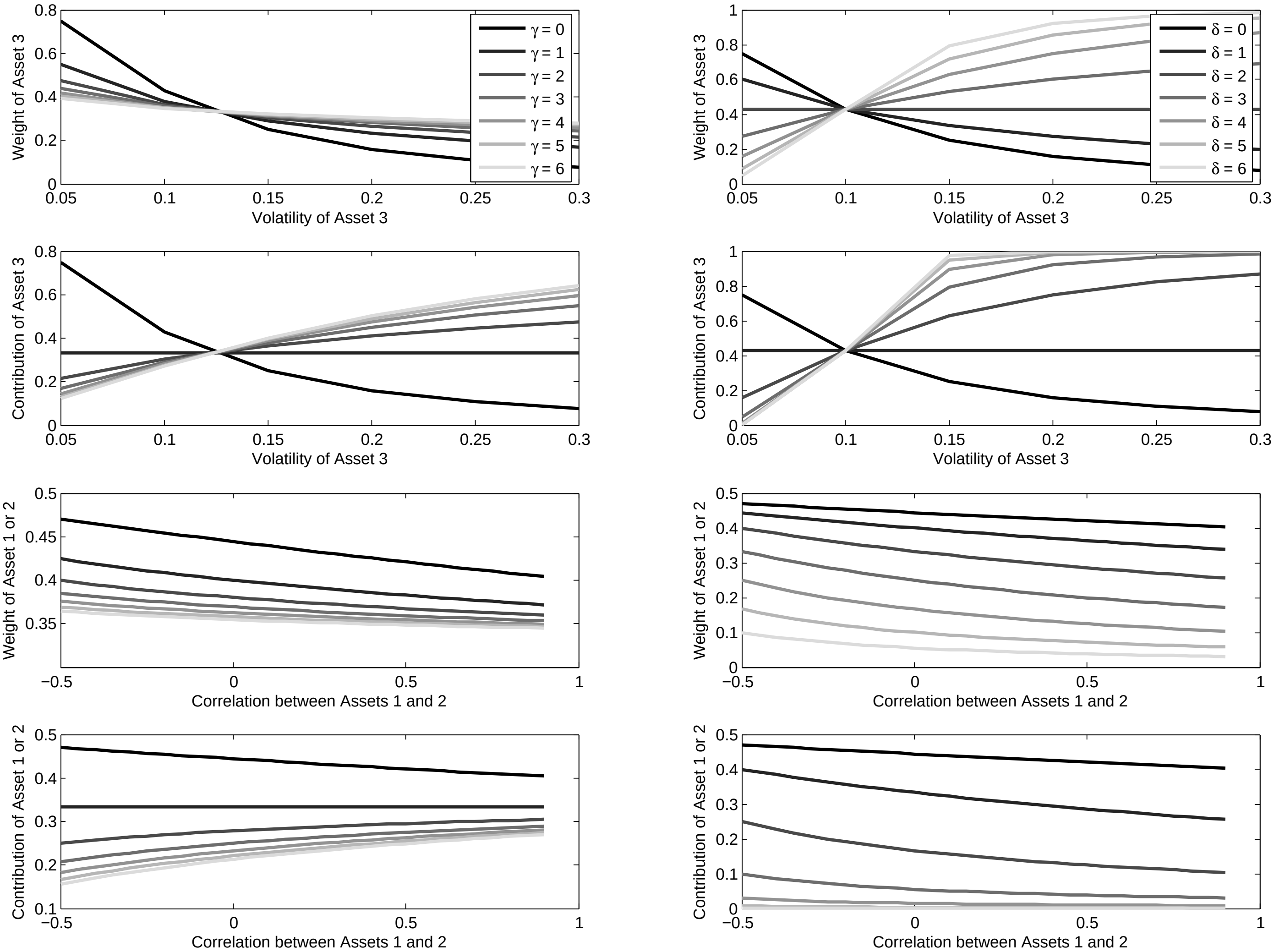

Figure 2: Sensitivity of the Optimal Risk-based Solutions to Volatility and Correlation

3.1.3 Risk-Based Portfolio Beta

The framework defined above is also helpful to investigate the general market sensitivity of the portfolio, β_{pm}. By definition of a beta, we have

$$\beta_{pm} = \frac{\text{cov}(R_m, R_p)}{\sigma_m^2},$$

where σ_m is the market volatility, R_p are the returns of the portfolio and R_m are the returns of the market. Since $R_m = \sum_{i=1}^n w_i^{(m)} R_i$ with $w_i^{(m)}$ being the weight of asset i in the market, we deduce that

$$\beta_{pm} = \sum_{i=1}^n w_i^{(m)} \frac{\text{cov}(R_i, R_p)}{\sigma_m^2}.$$

Using the fact that $\text{cov}(R_i, R_p) = \beta_{ip}\sigma_p^2$, we finally get

$$\beta_{pm} = \frac{\sigma_p^2}{\sigma_m^2} \sum_{i=1}^n w_i^{(m)} \beta_{ip}. \tag{15}$$

Inserting in (15) the individual beta properties listed in Table 1 leads to determine the expression for the beta of the characteristic risk-based portfolios. Starting with MV, we immediately deduce that

$$\beta_{pm}^{(\text{MV})} = \frac{\sigma_{(\text{MV})}^2}{\sigma_m^2}. \tag{16}$$

The beta of the MV portfolio is only given by the ratio between its own variance and the market variance. As no other portfolio can lead to a lower variance than MV by definition, it is immediate to see that its (ex-ante) beta will be below 1 at any point in time; the sole case where it will be equal to 1 being when MV is the market itself.

In the case of MD, we get

$$\beta_{pm}^{(\text{MD})} = \frac{\sigma_{(\text{MD})}^2}{\sigma_m^2} \times \frac{\sum_{i=1}^n w_i^{(m)}\sigma_i}{\sum_{i=1}^n w_i\sigma_i}. \tag{17}$$

Introducing diversification ratios of MD and of the market, $DR_{(\text{MD})} = \frac{\sum_i w_i\sigma_i}{\sigma_{\text{MD}}}$ and $DR_m = \frac{\sum_i w_i^{(m)}\sigma_i}{\sigma_m}$, we can rewrite equation (17) as

$$\beta_{pm}^{(\text{MD})} = \frac{\sigma_{(\text{MD})}}{\sigma_m} \times \frac{DR_m}{DR_{(\text{MD})}}. \tag{18}$$

As $DR_{(\text{MD})}$ is necessarily higher to DR_m since the MD portfolio is maximizing diversification ratio, we deduce that a sufficient, but not necessary, condition for MD having a beta below 1 is that its own volatility is below the one of the market. As MD portfolio tends to favor low-beta assets, as we shall see later, this feature is likely. More generally, if the market is fairly concentrated in a few volatile assets, the reduction of beta achieved by MD might be very significant.

For RP, we get

$$\beta_{pm}^{(\mathrm{RP})} = \frac{\sigma^2_{(\mathrm{RP})}}{\sigma^2_m} \times \frac{1}{n} \times \sum_{i=1}^{n} \frac{w_i^{(m)}}{w_i}. \tag{19}$$

It is more complex to evaluate this condition. However, we can make some reasonable approximations. In particular, for a homogenous market in terms of risk, we have $w_i \approx n^{-1}$, in which case, whatever the market structure $w_i^{(m)}$, we get $\beta_{pm}^{(\mathrm{RP})} = \frac{\sigma^2_{(\mathrm{RP})}}{\sigma^2_m}$. This implies that RP will have a beta below 1 if its volatility is below the one of the market. This is probable as RP also tends to prefer low-beta assets, as we will show later. However, the reduction in beta will be lower than the one achieved by MV and probably lower than the one of MD.

Finally, for EW, we deduce that

$$\beta_{pm}^{(\mathrm{EW})} = \frac{\sigma^2_{(\mathrm{EW})}}{\sigma^2_m} \sum_{i=1}^{n} w_i^{(m)} \times \frac{n \times \sum_{j=1}^{n} \sigma_{ij}}{\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij}}. \tag{20}$$

The value of the beta is highly dependent on the structure of the market in that case. If the market tends to be concentrated in assets with low (high) volatility or low (high) correlation with other assets, then the beta of the EW will be probably superior (inferior) to one.

3.2 Special Covariance Cases

Up to now, we have shown that risk-based strategies can be distinguished on the basis of two key parameters, notably regarding their sensitivity to risk parameters. These results were obtained in a general case, i.e. without any specific hypotheses on the variance-covariance matrix. We now give further insights on these results in the case of particular variance-covariance specifications [15].

3.2.1 Unique Correlation

We saw that a closed-form solution is not available in the general case for risk-based allocation portfolios. This feature had already been observed for RP portfolios but it has been shown in that case that, for a unique correlation $\rho_{ij} = \rho$, optimal solution is defined as $w_i^* = \sigma_i^{-1} / \sum_k^n \sigma_k^{-1}$ (Maillard, Roncalli, and Teiletche, 2010). This result is widely used in practice, as most risk

[15] Using specific correlation structures for portfolio selection has been advocated for a long time in the academic literature, starting with Elton and Gruber (1973).

parity providers are defining their portfolios based on the inverse individual volatilities, ignoring correlation[16].

A very close result can be obtained for MD. When correlation is unique, $\rho_i = \rho\ \forall i$, the marginal risk contributions can be rewritten as $MRC_i = (1-\rho)w_i\sigma_i^2 + \rho\sum_{k=1}^n w_k\sigma_k\sigma_i$. Plugging these values into (2) for $(\gamma, \delta) = (0, 1)$ leads to

$$(1-\rho)w_i\sigma_i + \rho\sum_{k=1}^{n} w_k\sigma_k = (1-\rho)w_j\sigma_j + \rho\sum_{k=1}^{n} w_k\sigma_k. \tag{21}$$

When $\rho < 1$, this condition becomes $w_i\sigma_i = w_j\sigma_j$, which, coupled with the budgetary constraint is equivalent to $w_i^* = \sigma_i^{-1}/\sum_k^n \sigma_k^{-1}$. Thus, when the correlation is unique and different from 1, MD and RP are exactly the same portfolios and both diversification measures are equivalent. Notice that for $\rho = 1$, the solution for (21) is not defined. The rationale for this result is that the diversification ratio is then equal to 1 whatever the portfolio composition, hence generating an infinity of solutions.

In the more global case of generalized risk-based portfolios, having a unique correlation is not enough to get a closed-form solution[17]. Hopefully, there are some instances where we can find some closed-formulas.

First, if $\gamma = 0$, such as for MV, we get a closed-form solution (see Appendix B). Observe that this result is not surprising as we already notice that there is no endogeneity in the solution when $\gamma = 0$ (see above). The solutions are characterized by decreasing weight in individual total risks, all the more as tolerance for individual total risk (δ) is low.

Second, we also obtain closed-form solutions for general γ specification under three specific constant correlation cases[18]: $\rho = -(n-1)^{-1}$, $\rho = 0$ and $\rho = 1$. When the constant correlation matrix reaches its lower bound[19], $\rho = -(n-1)^{-1}$, risk-based portfolios are all defined such that the optimal weights are given by

$$w_i^* = \sigma_i^{-1}/\sum_{k=1}^{n}\sigma_k^{-1}.$$

[16]Chaves, Hsu, Li, and Shakernia (2012) quote this approach as "naive risk parity".

[17]Similar issues arise in the (less relevant) case where volatility is unique, $\sigma_i = \sigma\ \forall i$. In that case, we have

$$w_i^* = \frac{\left(\sum_k w_k\rho_{ik}\right)^{-\frac{1}{\gamma}}}{\sum_j\left(\sum_k w_k\rho_{jk}\right)^{-\frac{1}{\gamma}}}.$$

This shows that more correlated assets are penalized. We also see that this solution is still endogenous and does not depend on δ, implying notably that MD and MV constitute the same portfolio.

[18]Bruder and Roncalli (2012) obtain a similar result for their risk-budgeting approach.

[19]This bound corresponds to the lowest possible constant correlation coefficient ensuring the positive definiteness of the correlation matrix.

This generalizes the result previously obtained for RP. We also have a closed-form expression when all assets are uncorrelated. The uncorrelated case is interesting because there is an increasing set of proposals in the asset management industry to apply risk-based strategies on (supposedly) uncorrelated factors. When all assets are uncorrelated, that is if $\rho_{i,j} = 0$ for $i, j = (1, 2, ..., n)$, the beta of the asset i with the optimal portfolio becomes $\beta_{ip} = \left(w_i\, \sigma_i^2\right)/\sigma_p^2$, which implies that

$$w_i^* = \frac{\sigma_i^{-\frac{2-\delta}{\gamma+1}}}{\sum_{k=1}^n \sigma_k^{-\frac{2-\delta}{\gamma+1}}}.$$

This expression sheds further light on the sensitivity to parameters γ and δ discussed in previous subsections. Indeed, it clearly shows that the MV strategy is more sensitive to changes in the asset risk parameters when compared to the RP and MD portfolios, since its weights are exactly proportional to the inverse of the asset return variances, i.e. $w_i^* \propto \sigma_i^{-2}$, while the weights of the RP and MD portfolios are both proportional to the inverse of the asset volatilities, i.e. $w_i^* \propto \sigma_i^{-1}$. Notice that when $\gamma = 0$ and $\delta = 2$, we obtain the EW portfolio. This implies that when assets are uncorrelated, MD constitutes a middle-ground between MV and EW, equivalently to RP.

Similarly, if we have perfect correlation, that is if $\rho_{i,j} = 1$ for $i, j = (1, 2, ..., n)$, the beta of the asset i with respect to the optimal portfolio then becomes $\beta_{ip} = \sigma_i \left(\sum_k w_k \sigma_k\right)/\sigma_p^2$ and it follows that

$$w_i^* = \frac{\sigma_i^{-\frac{1-\delta}{\gamma}}}{\sum_{k=1}^n \sigma_k^{-\frac{1-\delta}{\gamma}}}.$$

For RP, the solution is not affected as it remains the same as for other unique correlation values. For MV, the solution becomes highly sensitive to risk parameters. If a long-only constraint applies, the MV invests 100% of its holdings in the asset with the lowest volatility. As already mentioned, the MD is not defined in the case of a unique correlation equal to 1.

3.2.2 Factor Models

We now investigate the properties of the generalized risk-based portfolios in a "CAPM" world, extending the results of Clarke, De Silva, and Thorley (2011, 2013) and Scherer (2011). Under the simplifying assumption of a single-factor risk model, the individual total risks and the pairwise covariances are given respectively by

$$\begin{cases} \sigma_i^2 = \beta_{im}^2\, \sigma_m^2 + \sigma_{\varepsilon_i}^2, \\ \sigma_{ij} = \beta_{im}\, \beta_{jm}\, \sigma_m^2, \end{cases}$$

where β_{km} and $\sigma^2_{\varepsilon_k}$ represent the market beta and idiosyncratic variance of asset k, respectively, and σ^2_m corresponds to the market variance (i.e. the factor we consider).

Using the matrix inversion lemma (see Woodbury, 1950), the optimal risk-based solution can be written as (see Appendix C for more details)

$$w_i = \kappa \left[\left(\frac{\sigma_i^\delta}{\sigma^2_{\varepsilon_i}} \right) \left(1 - \frac{\beta_{im}/\beta_{ip}}{\beta_U} \right) \right]^{\frac{1}{\gamma+1}}, \tag{22}$$

with

$$\beta_U = \frac{\frac{1}{\sigma^2_m} + \sum_{k=1}^{n} \frac{\beta^2_{km}}{\sigma^2_{\varepsilon_k}}}{\sum_{k=1}^{n} \frac{\beta_{km}}{\sigma^2_{\varepsilon k}} \beta_{kp}},$$

where κ is a budgetary normalization constant, β_{ip} corresponds to the individual asset beta with respect to the risk-based portfolio, and β_U is the Unconstrained threshold beta.

While equation (22) verifies the positive weight constraint for γ being odd so that the RP portfolio is long-only by definition, the unconstrained risk-based versions will generally involve long and short positions for assets with betas lower and greater than β_U respectively. In specific cases, it is possible to obtain explicit long-only solutions. Indeed, when $\gamma = 0$, the optimal weights in the constrained risk-driven portfolios are given by

$$w_i = \left(\kappa \frac{\sigma_i^\delta}{\sigma^2_{\varepsilon_i}} \right) \left(1 - \frac{\beta_{im}/\sigma_i^\delta}{\beta_L} \right) \text{ for } \frac{\beta_{im}}{\sigma_i^\delta} < \beta_L, \text{ else } w_i = 0, \tag{23}$$

with

$$\beta_L = \frac{\frac{1}{\sigma^2_m} + \sum_{\beta_{im}/\sigma_i^\delta < \beta_L} \frac{\beta^2_{km}}{\sigma^2_{\varepsilon_k}}}{\sum^{n}_{\beta_{im}/\sigma_i^\delta < \beta_L} \frac{\beta_{km}\sigma_k^\delta}{\sigma^2_{\varepsilon k}}}$$

where $\beta_{km}/\sigma_k^\delta$ corresponds to the individual risk-adjusted beta and β_L represents the Long-only threshold beta that cannot be exceeded in order for an asset to be in the constrained optimal long-only solution.

Expressions (22) and (23) show that, at least for low δ, risk-based portfolios will have a clear preference for low-beta and low-volatility stocks, therefore capturing well-known anomalies[20]. The preference for low-volatility and low-beta is all the more important as γ is small. MV will

[20] These anomalies have been extensively documented in the literature. See, for recent evidence, Baker, Bradley, and Wurgler (2011), Baker and Haugen (2012), Blitz and Vliet (2007) or Asness, Frazzini, and Pedersen (2012), as well as section 4.3 of current paper.

thus have a much stronger preference for low-beta and low-volatility than RP, and even more than EW who has no interest in it.

The role of δ is more complex. To see this, consider the two following extreme cases: one where assets have no exposure to the common risk-factor, and the other one where they have no idiosyncratic risk. In the first case, where $\beta_{im} = 0$, we immediately deduce from (22) that

$$w_i \propto (\sigma_{\varepsilon_i})^{-\frac{2-\delta}{1+\gamma}}. \tag{24}$$

In the second case where $\sigma^2_{\varepsilon_i} = 0$, the individual total risk simplifies to $\sigma_i = \beta_{im}\sigma_m$. Combining this with the expression of marginal contributions in a factor model, $MRC_i = \frac{\beta_{im}\sigma^2_m\left(\sum_k w_k\beta_{km}\right)}{\sigma_p}$, and plugging these values into (2), the optimal risk-based solution (13) then reads as

$$w_i \propto (\beta_{im})^{-\frac{1-\delta}{\gamma}}. \tag{25}$$

Equations (24) and (25) illustrate the ambiguous role of δ. In the first case, risk-based weights are decreasing functions of the idiosyncratic risks up to $\delta = 2$, where portfolios become increasingly invested in higher idiosyncratic risks. In the second case, we see that risk-based weights are decreasing functions of the individual betas for $0 \leq \delta \leq 1$, but becomes increasing functions of individual betas for $\delta > 1$. Eventually, when $\delta \to \infty$, we end up with a portfolio being fully invested in the asset having the highest beta or the highest idiosyncratic risk in specific cases, and the highest total risk in the general case.

We conclude this analysis of generalized risk-based models with factor models with three additional remarks. First, we can observe that our single-factor specification encompasses the ones which have been obtained by Scherer (2011), Clarke, De Silva, and Thorley (2011, 2013). For the RP solution, a key difference is that our generalized approach leads to a more simple expression than in Clarke, De Silva, and Thorley (2013), with

$$w_i^{(\mathrm{RP})} \propto \left[\left(\frac{n^{-1}}{\sigma^2_{\varepsilon_i}}\right)\left(1 - \frac{\beta_{im}/\beta_{ip}^{(\mathrm{RP})}}{\beta_U^{(\mathrm{RP})}}\right)\right]^{\frac{1}{2}}.$$

Second, it is straightforward to extend the formulas (22) and (23) to the case of a multi-factor model, by just rewriting the individual total risks and covariance terms as functions of several factors and plugging these expressions into (2) along the same lines as for the one-factor model. Naturally, the interpretation in a multi-factor setting is more complex. Third, the factor-model approach naturally leads to the definition of iterative algorithms for efficiently getting general risk-based portfolio weights, as described in Appendix D.

As a global summary of this analytical section, it clearly appears that risk-based portfolios have to be differentiated between the case where $\gamma = 0$, that is notably MV and MD [21], and when $\gamma \neq 0$. As we have discussed, a zero-γ restriction allows to get non-endogenous solutions and to fulfill the duplication invariance property. On the downside, however, the higher sensitivity of the zero-γ portfolios to estimation noise naturally leads to much higher instability in solutions and turnover. Furthermore, if long-only restrictions apply, it leads them to be much more concentrated in capital. Without such restrictions, high-beta or high-volatility assets would be shorted. By imposing short-sales restrictions, those assets will not be present anymore in the portfolio. This is particularly the case for MV. At the opposite, RP or EW will hold all the assets of the universe.

4 Empirical Illustrations

In this section, we build risk-balanced portfolios on the international developed markets equity universe, according to our modified risk contribution formula, and investigate their properties. Our goal here is two-fold: demonstrate the capacity of our approach to deal with large-scale portfolio problems, and illustrate the impact of the regularization and risk tolerance parameters on the risk-driven allocations in a "real-life"context. To this end, we investigate the ex-ante characteristics of the risk-based solutions when continuously varying the γ and δ parameters and check the sensitivities and biases of the corresponding portfolios. We do not intent here to demonstrate the superiority of one set of parameters over another, just to illustrate what they implicitly mean in terms of portfolio choices. We find that the risk-based equity portfolios respond to the variation of the parameters in accordance with our theoretical expectations.

4.1 Data and Empirical Solutions

We evaluate the risk-based strategies for the MSCI World index constituents, from January 2002 to October 2012. We use weekly USD returns and rebalance our strategies at a monthly frequency. For a given set of parameters, the optimal weights of the risk-based strategies depend only on an estimate of the variance-covariance matrix. The problem of getting a robust estimate of a large covariance matrix given a limited time-series sample is well-known, and to cope with it we use a single-factor risk model to estimate the ex-ante covariance matrix[22]. At

[21] To be more precise, we shall also distinguish portfolios where $\gamma = 0$ but $\delta \geq 1$. In this case, we have portfolios where the increasing risk tolerance allowed by higher δ can lead to very risky portfolios and they can probably not being qualified as "risk-based" anymore.

[22] The illustrative results presented below are thus contingent on that particular risk model, and shall probably not hold against all alternatives. However, the fact that they confirm our analytical results makes us confident in their robustness.

each rebalancing date, we update the risk model parameters, from a two-year rolling window of individual security returns of the most recent historical data. We keep only stocks in the MSCI World index for which there are 2 years of returns available at the date of analysis, and regress them against the market index returns. This leads us to an average sample size of 1630 stocks per month. Following Clarke, De Silva, and Thorley (2011, 2013), the historical betas are adjusted by half towards 1 each month. Negative betas are then set to 0. Similarly, the empirical log idiosyncratic risks are adjusted by a third towards the cross-sectional average log idiosyncratic risk for that month. After regularization[23], we observe that the individual idiosyncratic risks go from 18% for the first decile to 46% for the last decile; likewise the inter-decile spread of the predicted betas is relatively wide, from 0.63 to 1.47, and the explanatory power of the single-factor model goes from virtually 0% to 65%. We then use the single-factor analytic solutions established in subsection 3.2.2 and Appendix D to obtain the individual compositions of the risk-optimized portfolios. Indeed, the single-factor framework yields an analytical formula which is either a closed-form (for $\gamma = 0$), or converges to a fixed-point (for $\gamma \neq 0$), each asset being identified by its risk-adjusted beta as in (22). These algorithms are almost instantaneous, as they do not demand a matrix inversion. This makes them the methodology of choice for an extensive exploration of the (γ, δ) plane, contrary to traditional numerical optimization techniques which turn out to be slow on large universes.

Figure 3 shows the individual security weights for various long-only risk-based portfolios as of October 2012. The weights (on a log-scale) are shown on the y-axis, sorted from highest to lowest. As the regularization parameter γ increases, we observe that the optimal weight distribution flattens, as the differences between the individual risk contributions contribute less to the optimization criterion[24]. In particular, the number of lines with zero weight decreases. The effect of increasing the risk tolerance parameter δ is on the contrary to concentrate the portfolio composition on the less correlated assets with highest individual total risks. Note that while the weight structure of the risk-based portfolios looks similar for $\gamma = 0$, whatever δ, the stocks selected are not identical.

4.2 Portfolios Risk Characteristics

To facilitate the interpretation of the risk-based equity investing strategies, we now investigate their aggregate risk characteristics over the full sample, for realistic ranges of γ and δ

[23] The regularization scheme is preferable with so many assets to facilitate computation and avoid extreme solutions that would result from estimation errors.

[24] The EW portfolio is not necessarily feasible by setting a high γ in the specific case where the equal weight distribution would yield at least one negative total risk contribution.

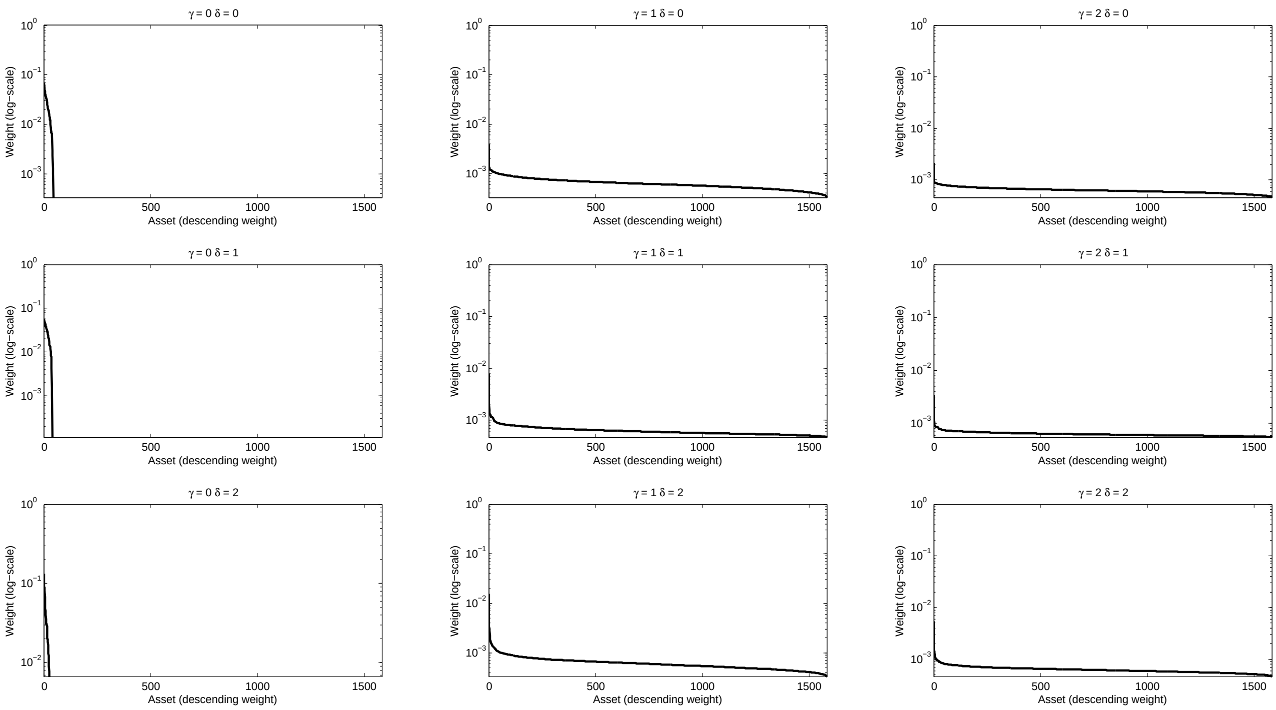

Figure 3: Risk-Based Long-Only Portfolio Weights for Chosen Values of γ and δ (MSCI World Universe, October 2012)

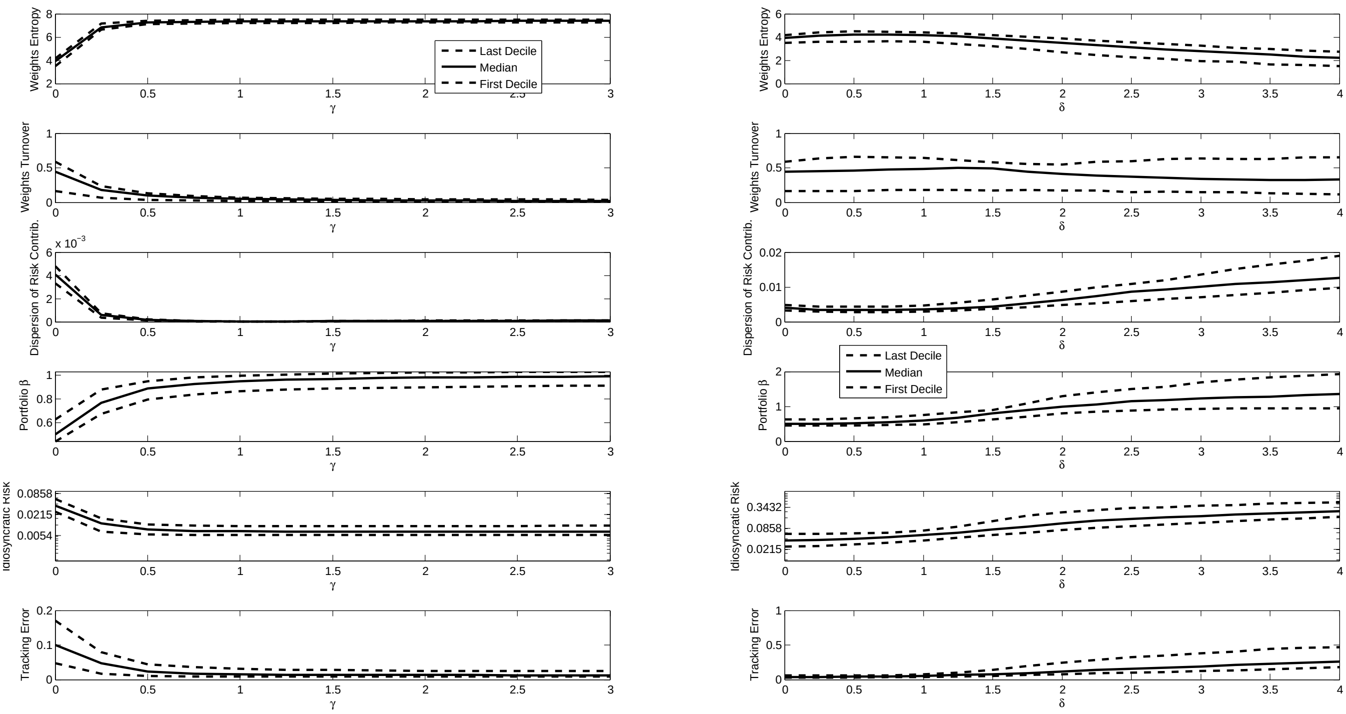

Figure 4: Ex-Ante Characteristics of Risk-Based Equity Strategies when Varying γ and δ (MSCI World Universe, January 2002-October 2012)

parameters. For that purpose, we report the median values for the portfolio entropy[25], the month-to-month portfolio turnover, the dispersion of the relative risk contributions, the portfolio's market β, the idiosyncratic risk and the tracking error with respect to the MSCI World index. We also report the first and last decile of the distribution of each characteristic over the whole 130 months sample.

Figure 4 represents the ex-ante characteristics of the risk-optimized portfolios when varying either γ or δ. On the left hand-side, we let γ vary between 0 to 3, while keeping δ fixed at 0. Unsurprisingly, the low-γ portfolios are the most concentrated, investing in the stocks with the lowest market sensitivity (lowest beta). As a consequence, their tracking errors with respect to the MSCI World index are relatively high. With a higher γ, the importance of the individual total risk estimates decreases and the resulting risk-based portfolios rely less on a small set of low-beta and low-volatility assets. We observe that the concentration of the risk contributions is minimal for $\gamma = 1$ (RP), declining strongly from $\gamma = 0$ and picking up slightly

[25] The Shannon entropy is a measure of randomness in information theory which can be used as a diversification measure in a portfolio context. For the portfolio weights $w_i, (i = 1, ..., n)$, $\sum_i^n w_i = 1$, the entropy is given by $H(w) = -\sum_i^n w_i \times \ln(w_i)$, and is maximized for $w_i = n^{-1}\ \forall i$.

above 1. The turnover of the risk-based portfolios declines with γ, since the EW portfolio has, by definition, no turnover (all stocks have the same weight whatever the date). Interestingly, the most dramatic impact of varying γ occurs between the MV and RP (γ between 0 and 1), while portfolios between RP and EW are largely comparable with respect to their ex-ante characteristics. However, note that for the variables depicted here, the range of characteristics spanned by the risk-based equity strategies is relatively narrow. On the right-hand side of Figure 4, we let δ vary from 0 to 4, while γ is kept constant at 0. As δ increases, the ex-ante portfolio risk characteristics are modified. Starting from the MV strategy, we observe that the portfolio concentration decreases (entropy passes through a maximum) and picks up again, as the preference for volatility drives the optimal solution towards high-risk assets. This can also be seen on the aggregate market sensitivity, with the β peaking for high values of δ, as does the idiosyncratic risk. Notice that the range of variation of the median β and median residual risk is much higher than in the left column (γ-axis), meaning that the risk-based portfolios are much more sensitive to the tuning of the risk tolerance coefficient than the regularization one, everything else equal. Still, the median portfolio concentration and turnover are largely insensitive to the value of δ: the risk-based allocations tend to have on the δ-axis a heavy rotation on the few stocks they select. Regarding the statistical significance of the median characteristic differences, we observe that the dispersion of the values over the whole historical sample remains quite large for most of the variables of interest, so that the significance of the variations is difficult to ascertain. The tracking error, dispersion of risk contributions and concentration of the zero-δ portfolios are however always lower than the ones of the zero-γ portfolios.

Figure 5 additionally displays the evolution of ex-ante risk characteristics over time for some remarkable portfolios: market-capitalization index (MSCI World index), long-only constrained MV, long-only constrained MD, RP and EW. The risk characteristics are the portfolio concentration, the market β, the portfolio volatility and the tracking error. We see, whatever the date, a clear dichotomy between the four risk-based portfolios. The first group, composed of MV and MD, consists in concentrated, low-beta, low absolute volatility and high tracking-error portfolios. The second group, made of EW and RP, corresponds to well-diversified, less beta-biased, low tracking-error portfolios.

Overall, the empirical results are consistent with the theoretical considerations of the previous section. In the next subsection, we pursue the analysis by investigating the recent criticism raised by some authors (Scherer, 2011; Amenc, Goltz, and Lodh, 2012) that risk-based portfolio strategies implicitly take active style bets, without specifically controlling the risk-reward optimality.

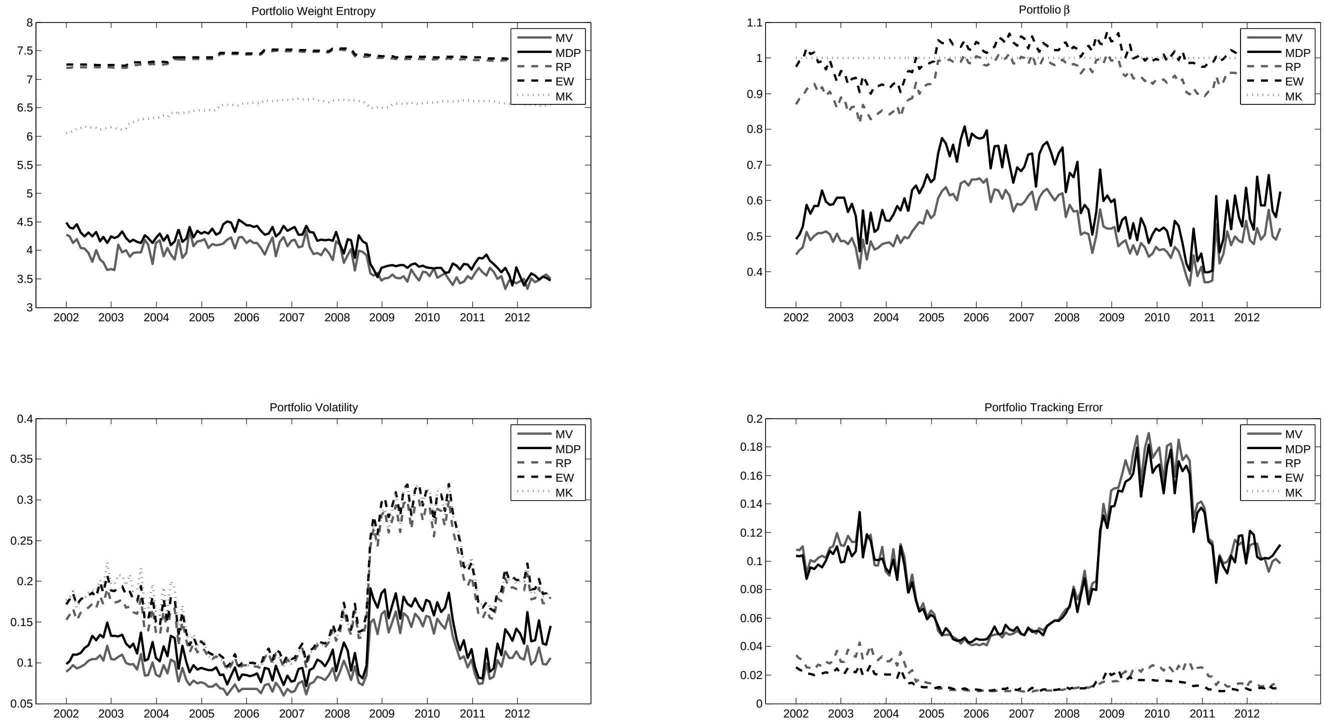

Figure 5: Evolution of the Ex-Ante Characteristics of Risk-Based Equity Strategies (MSCI World Universe, January 2002-October 2012)

4.3 Style Biases of the Risk-Based Portfolios

We now investigate these criticisms by examining the style exposures of our allocations. As we have constructed the covariance matrix from the single factor βs, the optimization does not use any other common dependency between the stocks, and thus does not control for any style bias that may arise. To see whether ignoring these other risk factors really impacts the portfolios, we adopt a framework consistent with Fama and French (1992). According to these authors, portfolio returns are not explained solely by their exposure (beta) to market risk, but also by additional influences related to stock capitalizations and valuations. The original size (SMB) and valuation (HML) factors have been completed by Carhart's momentum factor (UMD) (Carhart, 1997), as another potential common source of excess returns. The size factor represents the effect of incorporating small capitalization stocks into the portfolios (thus getting an illiquidity premium), valuation means taking cheaper stocks according to book-to-price (the premium can be deemed as a reward for the correlation with the business cycle), while momentum means over-weighting stocks whose past performances have been good (various behavorial explanations such as herd behavior have been used to justify this). Additionally, we define both a low-residual volatility mimicking factor portfolio (LV) and a low β factor (LB), along the lines of Scherer (2011) and Leote, Lu, and Moulin (2012). A general justification for the low-risk anomaly is that institutional restrictions prevent investors to leverage their optimal portfolio as much as necessary to achieve their target return, thus inciting them to use high volatility assets instead, leading to demand distortions and better risk-adjusted performance of low-risk assets (see, for instance, Frazzini and Pedersen, 2012).

Since the factors available on Kenneth French's site are only defined for the US equity market, we duplicate their methodology on the MSCI World universe. Building raw long-short portfolios by sorting stocks on each variable could yield correlated factors. This is the case traditionally for valuation and size, and for our new factors on residual volatility and beta. To neutralize these correlations, we use double-sorting, as in the the original Fama-French algorithm. For each style risk factor considered, we first sort on the correlated variable and then build long-short portfolios conditional on the quintiles of the correlated variable, by being long the first quintile and short the last quintile (thus, ignoring the 60% of stocks in the middle of the distribution), with stocks equal-weighted within these baskets.

For instance, to get the SMB factor, we first separate the stocks by valuation, then in each valuation quintile build long-short portfolios on size (smaller size quintile minus larger size quintile). The SMB factor is the final simple average of these valuation-conditional long-short portfolios. The HML factor uses the size as the conditioning variable. We also proceed in the same way for the LV and LB factors, making each one conditional on the other, with the addition that the long-short portfolios are set up by by controlling for the beta ratio between the long and short legs, so that the allocation is by construction market-neutral on an ex-ante basis, similarly to Scherer (2011) or Leote, Lu, and Moulin (2012). That way, the effects on returns of low-beta and low-residual volatility are disentangled and the custom LV factor only reflects the latter. In the same way, we produce a market-neutral low-beta factor (LB) that is uncorrelated with the low-residual risk factor. The MOM factor is built as a long-short portfolio using the distribution of one-year total returns, being long the stocks in the first decile and short the ones in the last decile. The returns of these factors are the regressors that we use to explain the ex-post performance of the generalized risk based portfolios. We set here the dependent variable of regression as the excess return over the market of risk-based portfolios, so that a low-beta strategy correspond to a negative Market coefficient in the estimation.

Figure 6 represents the portfolios regression β on the 6 style factors we have defined (from top to bottom Market, SMB, HML, MOM, LV, LB), when varying γ (keeping δ equal to 0; left column) and δ (keeping γ equal to 0; right column). The coefficients are shown within their one-percent bounds with normal standard-errors. The total number of observations is 129, corresponding to the monthly portfolio returns from February 2002 to October 2012. As γ

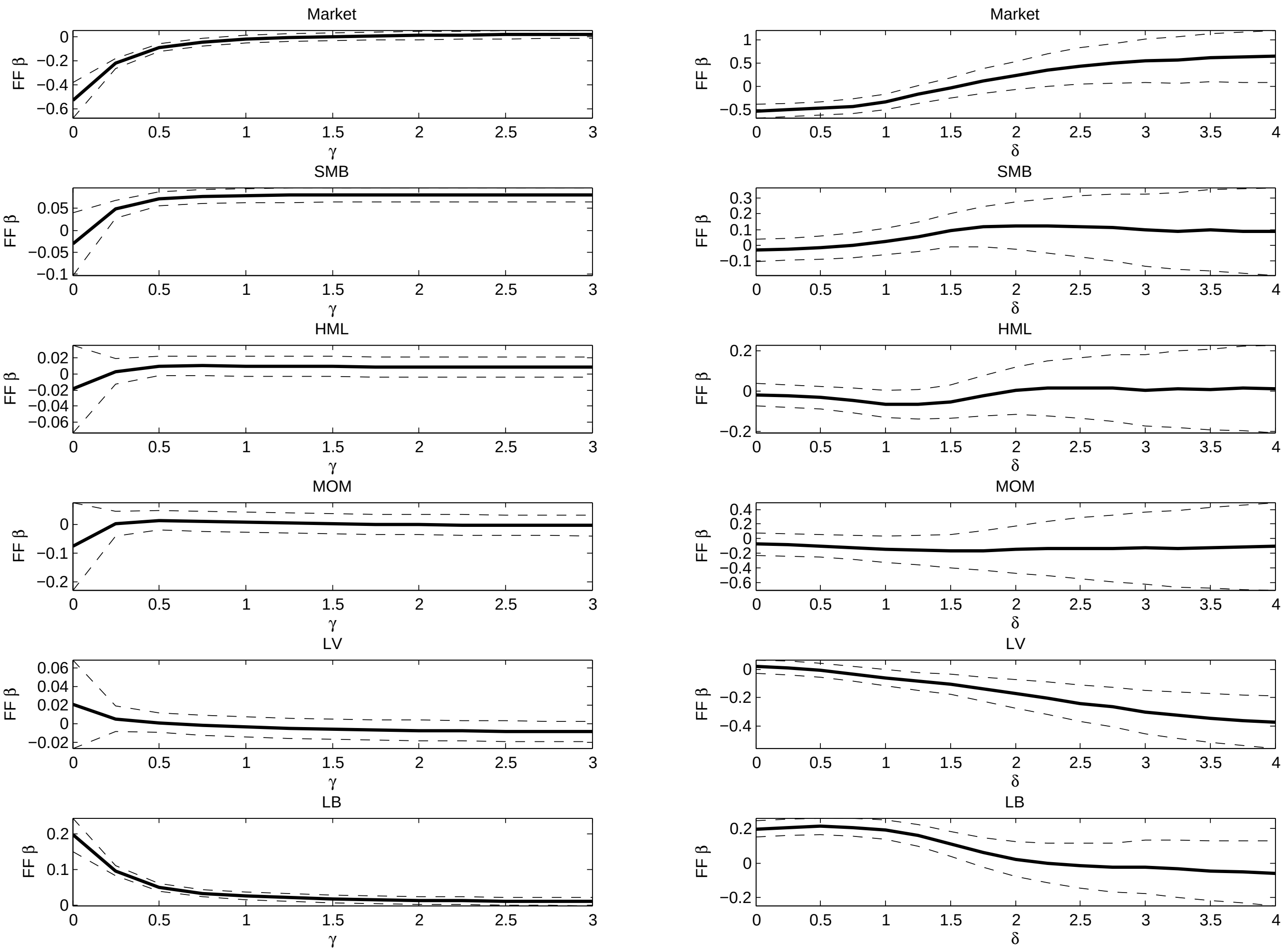

Figure 6: Ex Post Style Exposures when Varying γ ($\delta = 0$) and δ ($\gamma = 0$) (MSCI World Universe, January 2002-October 2012

increases (left column), the market beta of the portfolios tend to neutrality, and they become positively exposed to the size factor. The low-β bias, maximal for the MV portfolio decreases to zero with increasing γ, though it is still significant for $\gamma = 1$ (RP). Varying δ affects the market exposure, the idiosyncratic risk and the low-beta bias. When the appetite for risk increases, the systematic risk of the portfolio increases, from its minimum for MV, as does the exposure to idiosyncratic risk, while the low-beta bias disappears. In this context, increasing δ also increases the instability of the coefficients, as can be seen from the widening of 1% bounds, which is a natural consequence of the concentration in highly volatile stocks. On both the δ and the γ axes, HML and MOM do not appear as very significant distinctive features among portfolios on average.

Table 2 details these results for the set of popular risk-based portfolios. MV has the lowest market beta of the set, as well as the most pronounced bias on low-beta stocks. MD has the same low-beta stocks bias, a slightly higher - but still significantly below one - market beta, and a negative exposure on the low-residual volatility factor: in this single factor context, maximizing the diversification, based on Choueifaty-Coignard (2008) measure, pushes both towards lower beta stocks and higher idiosyncratic volatility ones. RP exhibits two main exposures, size and low-beta, confirming its role as intermediary portfolio between MV and EW. It is also much more neutral in terms of exposure to the trend in the market (Market beta not significantly different from one) and, overall, seems the most balanced in terms of exposures to factors as already shown in other empirical works (see, e.g., de Leote, Lu, and Moulin, 2012). EW is biased towards small caps, but also shows both a market beta slightly above one and a negative bias on low residual volatility stocks. The fact that RP has a lower exposure than EW to SMB is natural if we assume that smaller stocks have higher individual total risks, which is a reasonable assumption as small-cap stocks tend to be less liquid and characteristic of smaller businesses[26]. By the same line of reasoning, it is not surprising to observe a negative size factor exposure for MV, even if not significant here.

To summarize, the empirical investigation of the risk-based portfolios yields very clear stylized facts, in this simplified single-factor framework. Increasing the regularization parameter γ leads to less concentrated equity portfolios, dampening the risk-based factors and mechanically increasing the exposure to the size factor as the weights are spread more evenly on all assets whatever their size (and individual beta). Increasing the risk tolerance parameter δ leads to an increase in the market beta as the allocation shifts towards riskier stocks, with an initial decrease in weight concentration and then an increase again when the appetite for risk leads to a portfolio focused on fewer assets[27]. The Fama-French ex-post regressions confirm the importance of the low-beta anomaly for all risk-based portfolios, while positive γ portfolios tend to have a small cap bias and positive δ portfolios a high idiosyncratic risk bias, in a CAPM framework. These characteristics are consistent with our theoretical expectations, confirming the validity of our factor-based approach to deliver fast and accurate large-scale portfolios.

[26]The Fama-French portfolios formed on size, available on Kenneth French's website, show that, from July 1926 to July 2012, the annualized volatility of monthly returns has been 35.3% for the low decile (10% of smallest stocks by size) vs 17.7% for the high decile (10% of largest stocks by size).

[27]Note that the risk-based portfolios will generally lie somewhere between these extremes, and respond continuously to variations of the two parameters. Thus the style biases inherent in the risk-based portfolios will generally not be so easily disentangled: changing a parameter will affect exposure to all factors simultaneously.

Table 2: Regression of the Returns of the Remarkable Portfolios on Style Factors, MSCI World, 2002-2012

	MV	MD	RP	EW
Market	-0.529**	-0.331**	-0.024	0.039**
	(-9.42)	(-4.97)	(-1.89)	(2.98)
SMB	-0.032	0.023	0.080*	0.082**
	(-1.15)	(0.69)	(12.61)	(12.56)
HML	-0.019	-0.063*	0.010*	0.007
	(-0.87)	(-2.50)	(2.07)	(1.40)
MOM	-0.076	-0.147*	0.008	-0.012
	(-1.28)	(-2.09)	(0.57)	(-0.84)
LV	0.021	-0.058**	-0.004	-0.012**
	(1.15)	(-2.65)	(-0.89)	(-2.71)
LB	0.197*	0.193**	0.026**	0.003
	(10.64)	(8.81)	(6.28)	(0.68)
R^2	0.78	0.62	0.79	0.79
DW p-value	0.81	0.82	0.51	0.15

Notes. t-statistics are given in parentheses. ** (respectively, *) indicates significance at the 1% (respectively, 5%) level.

5 Conclusion

In this article, we have proposed a generic mathematical definition of risk-based investing. We have shown that any risk-based portfolio can be obtained intuitively as a particular risk-balancing strategy, equalizing modified risk contributions with respect to two specific parameters: a regularization parameter and a risk tolerance coefficient. By varying these parameters, we obtain the general set of risk-based methodologies, of which the MV, MD, RP, and EW portfolios are particular elements. We have derived analytical solutions for special risk structures, such as a constant correlation structure and a single-factor model structure. We have established the general mean-variance optimality conditions. Besides providing intuition on the economic nature of risk-based investing, our analytical formulas also provide simple numerical algorithms for risk-based portfolio construction on large investment universes. Empirical results on the MSCI World universe also tend to confirm the findings of Scherer (2011) and Leote, Lu, and Moulin (2012), that the portfolio construction behind risk-based investing implicitly picks up asset pricing anomalies, especially the size and the low-beta pricing anomalies (see Fama and French, 1992, and Frazzini and Pedersen, 2011). In Table 3, we summarize the characteristics of the most popular risk-based strategies that has been investigated in this paper. Due to these differences, it is probably fair to say that no approach should be considered as unanimously superior to the others, for all market environments or type of investors. A potential interesting extension to this work could be to investigate the behavior of risk-based investing methods with respect to alternative allocation models such as fundamental investing.

Table 3: Risk-Based Investing Synthesis Table

Portfolio (γ, δ)	Strategy definition	Optimality conditions	Capital/risk distribution	Risk characteristics
MV $(0, 0)$	Equalizes MRC_i	Identical excess returns	Highly concentrated Highly sensitive to VCV High Turnover	Lowest risk Lowest beta ($\beta \ll 1$) High TE
MD $(0, 1)$	Equalizes vol-scaled MRC_i	Identical Sharpe ratios	Highly concentrated Highly sensitive to correls Moderately sensitive to vols High Turnover	Low/Medium risk Low beta ($\beta < 1$) High TE
RP $(1, 0)$	Equalizes TRC_i	Identical Sharpe ratios Unique correlation	Diversified in risk Moderately sensitive to VCV Medium Turnover	Medium risk β below but close to 1 Medium TE
EW $(\infty, 0)$	Equalizes w_i	Identical excess returns Identical volatilities Unique correlation	Diversified in capital Insensitive to VCV Low Turnover	Medium to High total risk Low/Medium specific risk β above but close to 1 Low/Medium TE

Notes: VCV denotes the covariance matrix, MRC_i marginal risk contributions, TRC_i total risk contributions, w_i i-th asset's weight, β portfolio's sensitivity to market return.

Appendices

Appendix A: Risk-Based Portfolio Mathematics

Using matrix notation, the risk-based equality condition, $w_i^{\gamma} \sigma_i^{-\delta} MCR_i = \tau \ \forall i$, can be restated as

$$\sigma_p^{-1} \, \mathbf{\Omega} \, \mathbf{w} = \tau \, \nu, \qquad \text{(A.A.1.)}$$

where $\nu = \left(\sigma_1^{\delta}/w_1^{\gamma}, ..., \sigma_n^{\delta}/w_n^{\gamma}\right)^{\mathbf{T}}$ is a column vector and τ is a constant factor. Premultiplying (A.A.1.) by $\mathbf{w^T}$ gives the analytical expression for the target constant, that is

$$\tau = \frac{\sigma_p}{\mathbf{w^T}\nu} = \frac{\sigma_p}{\sum_{k=1}^{n} w_k^{1-\gamma} \sigma_k^{\delta}}. \qquad \text{(A.A.2.)}$$

Substituting for the value of τ in (A.A.1.) and premultiplying once again by $\mathbf{\Omega}^{-1}$, we obtain

$$\mathbf{w} = \left(\frac{\sigma_p^2}{\mathbf{w^T}\nu}\right) \, \mathbf{\Omega}^{-1} \, \nu. \qquad \text{(A.A.3.)}$$

Imposing the budgetary condition, $\mathbf{1^T} \, \mathbf{w} = 1$, leads to the following solution for problem (2)

$$\mathbf{w} = \frac{\mathbf{\Omega}^{-1} \, \nu}{\mathbf{1^T}\mathbf{\Omega}^{-1} \, \nu}. \qquad \text{(A.A.4.)}$$

In general, (A.A.4.) does not lead to a closed-form solution as portfolio's weights are part of both sides of the equation. However, a closed-form solution does exist when $\gamma = 0$, as the risk-based solution can be expressed as

$$\mathbf{w} = \frac{\mathbf{\Omega}^{-1}\sigma^{\delta}}{\mathbf{1^T}\mathbf{\Omega}^{-1}\sigma^{\delta}}, \qquad \text{(A.A.5.)}$$

where $\sigma^{\delta} = \left(\sigma_1^{\delta}, ..., \sigma_n^{\delta}\right)^{\mathbf{T}}$ is a column vector.

Appendix B: Constant Correlation Analytical Expressions

In the case of a positive semi-definite constant correlation matrix, with generic term $\rho_{ij} = \rho$ if $i \neq j$ and $-(n-1)^{-1} \leq \rho < 1$, we have

$$\mathbf{\Omega} = \sigma\,\sigma^T \circ \mathbf{C}, \tag{A.B.1.}$$

where $\sigma = (\sigma_1, ..., \sigma_n)^{\mathbf{T}}$ is the column vector of individual volatilities, $\mathbf{C}$ is the constant correlation matrix, and $\circ$ is the Hadamard product symbol. On that basis, the inverse is given by

$$\mathbf{\Omega}^{-1} = \mathbf{\Gamma} \circ \mathbf{C}^{-1}, \tag{A.B.2.}$$

with

$$\mathbf{C}^{-1} = \frac{[(1-n)\,\rho - 1]\;\mathbf{I} + \rho\,\mathbf{1}\mathbf{1}^{\mathbf{T}}}{(n-1)\,\rho^2 - (n-2)\,\rho - 1},$$

where $\mathbf{1}$ is a unitary vector, $\mathbf{\Gamma}$ is a matrix with generic term $\Gamma_{ij} = 1/\left(\sigma_i\,\sigma_j\right)$ and $\mathbf{I}$ is the identity matrix. Replacing $\mathbf{\Omega}^{-1}$ by its expression (A.B.2.) into (A.A.4.), leads to

$$\mathbf{w} = \frac{[(1-n)\,\rho - 1]\;diag\left(\nu \circ \sigma^{-2}\right) + \rho\,\left(\sigma^{-1}\right)\left(\sigma^{-1}\right)^{\mathrm{T}}\nu}{\mathbf{1}^{\mathrm{T}}\left([(1-n)\,\rho - 1]\;diag\left(\nu \circ \sigma^{-2}\right) + \rho\,\left(\sigma^{-1}\right)\left(\sigma^{-1}\right)^{\mathrm{T}}\nu\right)}, \tag{A.B.3.}$$

where $\nu = \left(\sigma_1^{\delta}/w_1^{\gamma}, ..., \sigma_n^{\delta}/w_n^{\gamma}\right)^{\mathbf{T}}$ and $\sigma^{-\alpha} = \left(1/\sigma_1^{\alpha}, ..., 1/\sigma_n^{\alpha}\right)^{\mathbf{T}}$ are column vectors.

Using equation (A.A.5.) leads to the following closed formula when $\gamma = 0$

$$\mathbf{w} = \frac{[(1-n)\,\rho - 1]\;diag\left(\sigma^{\delta-2}\right) + \rho\,\left(\sigma^{-1}\right)\left(\sigma^{-1}\right)^{\mathrm{T}}\sigma^{\delta}}{\mathbf{1}^{\mathrm{T}}\left([(1-n)\,\rho - 1]\;diag\left(\sigma^{\delta-2}\right) + \rho\,\left(\sigma^{-1}\right)\left(\sigma^{-1}\right)^{\mathrm{T}}\sigma^{\delta}\right)}. \tag{A.B.4.}$$

where $\sigma^{\beta} = \left(\sigma_1^{\beta}, ..., \sigma_n^{\beta}\right)^{\mathbf{T}}$. We also note that when $\rho = -(n-1)^{-1}$ and $\gamma < +\infty$, the solution (A.B.3.) becomes

$$\mathbf{w} = \frac{diag\left(\sigma^{-1}\right)}{\mathbf{1}^{\mathrm{T}}\;diag\left(\sigma^{-1}\right)}. \tag{A.B.5.}$$

This solution corresponds exactly to the RP solution in the case of constant correlation (see Maillard, Roncalli, and Teiletche, 2010). This means that all the risk-based portfolios in the finite (γ, δ) space coincide with the RP portfolio when the unique correlation coefficient stands at its lowest possible value.

Appendix C: Single-Factor Expressions

Under the simplifying assumption of a single-factor risk model, the covariance matrix, $\mathbf{\Omega}$, can be decomposed as

$$\mathbf{\Omega} = \beta\,\beta^{\mathbf{T}}\,\sigma_m^2 + diag\left(\sigma_\varepsilon^2\right), \tag{A.C.1.}$$

where $\beta = (\beta_{1m}, ..., \beta_{nm})^{\mathbf{T}}$ is the vector of individual betas, σ_m^2 is the market variance, and $\sigma_\varepsilon^2 = \left(\sigma_{\varepsilon_1}^2, ..., \sigma_{\varepsilon_n}^2\right)^{\mathbf{T}}$ is the vector of residual variances. Using the Sherman-Morrison-Woodbury formula, the inverse of $\mathbf{\Omega}$ is given by

$$\mathbf{\Omega}^{-1} = diag\left(\sigma_\varepsilon^{-2}\right) - \frac{\left(\beta/\sigma_\varepsilon^2\right)\left(\beta/\sigma_\varepsilon^2\right)^{\mathbf{T}}}{\frac{1}{\sigma_m^2} + \left(\beta/\sigma_\varepsilon^2\right)^{\mathbf{T}}\beta}, \tag{A.C.2.}$$

where $\beta/\sigma_\varepsilon^2 = \left(\beta_{1m}/\sigma_{\varepsilon_1}^2, ..., \beta_{nm}/\sigma_{\varepsilon_n}^2\right)^{\mathbf{T}}$ represents the vector of idiosyncratic risk-adjusted betas. Substituting (A.C.2.) into (A.A.3) leads to

$$\mathbf{w} = \left(\frac{\sigma_p^2}{\mathbf{w}^{\mathbf{T}}\nu}\right)\left(diag\left(\sigma_\varepsilon^{-2}\right)\,\nu - \frac{\sum_{k=1}^{n}\frac{\beta_{km}}{\sigma_{\varepsilon k}^2}\frac{\sigma_k^\delta}{w_k^\gamma}}{\frac{1}{\sigma_m^2} + \sum_{k=1}^{n}\frac{\beta_{km}^2}{\sigma_{\varepsilon_k}^2}}\left(\beta/\sigma_\varepsilon^2\right)\right). \tag{A.C.3.}$$

That is

$$w_i = \left(\frac{\sigma_p^2}{\sum_{k=1}^{n} w_k^{\,1-\gamma}\,\sigma_k^\delta}\right)\left[\frac{\sigma_i^\delta}{w_i^\gamma\sigma_{\varepsilon_i}^2} - \frac{\beta_{im}}{\sigma_{\varepsilon_i}^2}\left(\frac{\sum_{k=1}^{n}\frac{\beta_{km}}{\sigma_{\varepsilon k}^2}\frac{\sigma_k^\delta}{w_k^\gamma}}{\frac{1}{\sigma_m^2} + \sum_{k=1}^{n}\frac{\beta_{km}^2}{\sigma_{\varepsilon_k}^2}}\right)\right]. \tag{A.C.4.}$$

Multiplying both sides of (A.C.4.) by w_i^γ and taking the $(\gamma+1)$-*th* root, we then deduce that

$$w_i = \kappa\left[\left(\frac{\sigma_i^\delta}{\sigma_{\varepsilon_i}^2}\right)\left(1 - \frac{\beta_{im}/\beta_{ip}}{\beta_U}\right)\right]^{\frac{1}{\gamma+1}}, \tag{A.C.5.}$$

with

$$\beta_U = \frac{\frac{1}{\sigma_m^2} + \sum_{k=1}^{n}\frac{\beta_{km}^2}{\sigma_{\varepsilon_k}^2}}{\sum_{k=1}^{n}\frac{\beta_{km}}{\sigma_{\varepsilon k}^2}\beta_{kp}},$$

where κ is a normalization constant and β_U is the Unconstrained threshold beta of the risk-based portfolio.

Turning to specific risk-based strategies, using the mathematical properties of the MV, MD, and RP portfolios (see Table 1), it follows that

$$\begin{cases} w_i^{(\mathrm{MV})} = \left[\frac{\sigma^2_{\mathrm{MV}}}{\sigma^2_{\varepsilon_i}}\right]\left(1 - \frac{\beta_{im}}{\beta_U^{(\mathrm{MV})}}\right), \\ w_i^{(\mathrm{MD})} = \left[\frac{\sigma^2_{\mathrm{MD}}}{\sum\limits_{k=1}^{n}\left(w_k^{(\mathrm{MD})}\sigma_k\right)}\frac{\sigma_i}{\sigma^2_{\varepsilon_i}}\right]\left(1 - \frac{\rho_{im}}{\rho_U^{(\mathrm{MD})}}\right), \\ w_i^{(\mathrm{RP})} = \left[n^{-1}\left(\frac{\sigma^2_{\mathrm{RP}}}{\sigma^2_{\varepsilon_i}}\right)\left(1 - \frac{\beta_{im}/\beta_{ip}^{(\mathrm{RP})}}{\beta_U^{(\mathrm{RP})}}\right)\right]^{\frac{1}{2}}, \end{cases} \tag{A.C.6.}$$

with $\beta_U^{(\mathrm{MV})} = \frac{\frac{1}{\sigma_m^2} + \sum\limits_{k=1}^{n}\frac{\beta^2_{km}}{\sigma^2_{\varepsilon_k}}}{\sum\limits_{k=1}^{n}\frac{\beta_{km}}{\sigma^2_{\varepsilon k}}}$, $\rho_U^{(\mathrm{MD})} = \frac{1 + \sum\limits_{k=1}^{n}\frac{\rho^2_{km}}{\left(1-\rho^2_{km}\right)}}{\sum\limits_{k=1}^{n}\frac{\rho_{km}}{\left(1-\rho^2_{km}\right)}}$ and $\beta_U^{(\mathrm{RP})} = \frac{\frac{1}{\sigma_m^2} + \sum\limits_{k=1}^{n}\frac{\beta^2_{km}}{\sigma^2_{\varepsilon_k}}}{\sum\limits_{k=1}^{n}\frac{\beta_{km}}{\sigma^2_{\varepsilon k}}\beta_{kp}^{(\mathrm{RP})}}$, and where we use the fact that $\sigma^2_{\varepsilon_k} = \sigma^2_{k}\left(1 - \rho^2_{km}\right)$ from the market model specification.

While RP is a long-only portfolio by construction, the functional form of the unconstrained MV and MD solutions (A.C.6) are preserved in the long-only constrained cases, with the Unconstrained threshold betas replaced by their following Long-only counterparts

$$\begin{cases} \beta_L^{(\mathrm{MV})} = \frac{\frac{1}{\sigma_m^2} + \sum\limits_{\beta_{km} < \beta_L^{(\mathrm{MV})}}\frac{\beta^2_{km}}{\sigma^2_{\varepsilon_k}}}{\sum\limits_{\beta_{km} < \beta_L^{(\mathrm{MV})}}\frac{\beta_{km}}{\sigma^2_{\varepsilon k}}}, \\ \rho_L^{(\mathrm{MD})} = \frac{1 + \sum\limits_{\rho_{km} < \rho_L^{(\mathrm{MD})}}\frac{\rho^2_{km}}{\left(1-\rho^2_{km}\right)}}{\sum\limits_{\rho_{km} < \rho_L^{(\mathrm{MD})}}^{n}\frac{\rho_{km}}{\left(1-\rho^2_{km}\right)}}. \end{cases} \tag{A.C.7.}$$

According to (A.C.6.) and (A.C.7.), individual assets need only to be sorted by their beta and correlation and then compared to the threshold parameter values $\beta_L^{(\mathrm{MV})}$ and $\rho_L^{(\mathrm{MD})}$ to determine which assets are kept in the long-only MV and MD solutions respectively.

Appendix D: Iterative Algorithms

Following the approach of Chaves, Hsu, Li, and Shakernia (2012), we derive three iterative methods for computing the weights of the general risk-based portfolios, namely a Newton algorithm and two power-based iterative methods that exploit the endogeneity of the solution of the general risk-based program (1). Notice that in the case where $\gamma = 0$, these algorithms are probably worthless as an analytical solution is readily available (see Appendix A). Still, one might argue that the iterative algorithms allow to avoid large matrix inversions (even if, in one-factor model, easier inversions can be obtained; see Appendix C).

The three following algorithms all have different strengths and weaknesses. Algorithm 1 is the most general, as it does not require assumptions on the risk model underlying the covariance matrix, but it gets notably slower as the size of the universe increases, and can have numerical scaling issues when the original starting points are not relevant. Algorithm 2 remains general, is much faster but is not always defined, such as for $\gamma = 0$ or $\beta_{ip} < 0$. Algorithm 3 is the fastest and the most robust but requires a restrictive assumption on the risk model. We use here a one-single factor model but our factor-based algorithm can be easily extended to a multi-factor setting. It also requires having components with positive β only.

Algorithm 1: Newton-Raphson's method

In order to derive a Newton algorithm for solving the general risk-based optimization problem, it is necessary to reexpress equation (A.A.1) such as

$$\mathbf{\Omega}\,\mathbf{w} = \varphi\,\nu,$$

where $\nu = \left(\sigma_1^\delta / w_1^\gamma, ..., \sigma_n^\delta / w_n^\gamma\right)^\mathbf{T}$ is a column vector and φ is a target constant. The Newton's iterative method for solving the non-linear system (2) can then be summarized as follows:

1. Set an integer counter j equal to 0. Obtain an initial guess for portfolio weights $\mathbf{w}^{(0)}$ (e.g. equal weights, market capitalizations or inverse volatilities), an arbitrary target constant $\varphi^{(0)}$ and a threshold η. Define $\mathbf{y}^{(0)} = \left(\mathbf{w}^{(0)}, \varphi^{(0)}\right)$.

2. Set $j = (j+1)$, and compute $\mathbf{y}^{(j+1)}$ such as

$$\mathbf{y}^{(j+1)} = \mathbf{y}^{(j)} - \left[\nabla F\left(\mathbf{y}^{(j)}\right)\right]^{-1}\; F\left(\mathbf{y}^{(j)}\right), \qquad \text{(A.D.1.)}$$

with

$$F\left(\mathbf{y}^{(j)}\right) = \begin{pmatrix} \mathbf{\Omega}\,\mathbf{w}^{(j)} - \varphi^{(j)}\;\nu^{(j)} \\ \mathbf{1}^\mathbf{T}\mathbf{w}^{(j)} - 1 \end{pmatrix},$$

and

$$\nabla F\left(\mathbf{y}^{(j)}\right) = \begin{pmatrix} \mathbf{\Omega} + \left(\gamma\,\varphi^{(j)}\right)\ diag\left(\sigma_1^{\delta}/\left(w_1^{(j)}\right)^{\gamma+1}, ..., \sigma_n^{\delta}/\left(w_n^{(j)}\right)^{\gamma+1}\right) & -\nu^{(j)} \\ \mathbf{1^T} & \mathbf{0} \end{pmatrix},$$

where $F\left(\mathbf{y}^{(j)}\right)$ is a system of $(n+1)$ nonlinear equations with $(n+1)$ unknowns $\left(\mathbf{w}^{(j)}, \varphi^{(j)}\right)$ and $\nabla F\left(\mathbf{y}^{(j)}\right) \in R^{(n+1)\times(n+1)}$ is the Jacobian matrix of $F\left(\mathbf{y}^{(j)}\right)$.

3. If the condition $\|\mathbf{y}^{(j+1)} - \mathbf{y}^{(j)}\| < \eta$ is fulfilled, then stop; otherwise return to step 2.

Algorithm 2: Power method with general beta expression

Recall that in the general case, the weights of a risk-based portfolio can be expressed as

$$w_i = \frac{\left(\sigma_i^{-\,\delta}\,\beta_{ip}\right)^{-\frac{1}{\gamma}}}{\sum_{k=1}^{n}\left(\sigma_k^{-\,\delta}\,\beta_{kp}\right)^{-\frac{1}{\gamma}}}. \tag{A.D.2.}$$

This solution is endogenous because the weight w_i is a function of the asset beta, β_{ip}, which in turn depends on the portfolio weight distribution, $\mathbf{w}$. While these dependencies preclude from finding simple analytical solutions for (A.A.3.), it also provides a simple iterative recipe for computing the portfolio weights for risk-based portfolios. The iterative process of the second algorithm can be summarized as follows:

1. Set an integer counter j equal to 0. Obtain an initial guess for portfolio weights $\mathbf{w}^{(0)}$ (e.g. equal weights, market capitalizations or inverse volatilities) and a threshold η.

2. Set $j = (j+1)$, compute the betas for all the individual assets, $\beta_{ip}^{(j)}$, with respect to the current portfolio $\mathbf{w}^{(j)}$, and then calculate the new weights as

$$w_i^{(j+1)} = \frac{\left(\sigma_i^{-\,\delta}\,\beta_{ip}^{(j)}\right)^{-\frac{1}{\gamma}}}{\sum_{k=1}^{n}\left(\sigma_k^{-\,\delta}\,\beta_{kp}^{(j)}\right)^{-\frac{1}{\gamma}}}. \tag{A.D.3.}$$

3. If the condition $\sqrt{\sum_{i=1}^{n}\left(w_i^{(j+1)} - w_i^{(j)}\right)^2} < \eta$ is fulfilled, then stop; otherwise return to step 2.

Algorithm 3: Power method with factor-based expression

Recall that under a single-factor model, the weights of a risk-based portfolio can be expressed as

$$w_i = \kappa\left[\left(\frac{\sigma_i^{\delta}}{\sigma_{\varepsilon_i}^2}\right)\left(1 - \frac{\beta_{im}/\beta_{ip}}{\beta_U}\right)\right]^{\frac{1}{\gamma+1}}, \tag{A.D.4.}$$

with

$$\beta_U = \frac{\frac{1}{\sigma_m^2} + \sum_{k=1}^{n} \frac{\beta_{km}^2}{\sigma_{\varepsilon_k}^2}}{\sum_{k=1}^{n} \frac{\beta_{km}}{\sigma_{\varepsilon k}^2} \beta_{kp}},$$

where σ_i and β_{ip} correspond respectively to the individual standard deviation and beta with respect to the risk-based portfolio, β_U is the Unconstrained threshold beta, and β_{im} and $\sigma_{\varepsilon_i}^2$ represent the market beta and idiosyncratic variance of asset i, respectively. The iterative process of the third algorithm can then be defined as:

1. Set an integer counter j equal to 0. Obtain an initial guess for portfolio weights $\mathbf{w}^{(0)}$ (e.g. equal weights, market capitalizations or inverse volatilities) and a threshold η.

2. Set $j = (j+1)$, compute the current portfolio parameters $\beta_{ip}^{(j)}$ and $\beta_U^{(j)}$ and calculate the new weights as

$$w_i^{(j+1)} = \left[\left(\frac{\sigma_i^\delta}{\sigma_{\varepsilon_i}^2} \right) \left(1 - \frac{\beta_{im}/\beta_{ip}^{(j)}}{\beta_U^{(j)}} \right) \right]^{\frac{1}{\gamma+1}}, \qquad \text{(A.D.5.)}$$

and rescale the weights to the predefined budget, e.g. $\sum_i w_i^{(j+1)} = 1$.

3. If the condition $\sqrt{\sum_{i=1}^{n} \left(w_i^{(j+1)} - w_i^{(j)} \right)^2} < \eta$ is fulfilled, then stop; otherwise return to step 2.

The same iterative process can be used in the presence of a short-sale constraint, by replacing in (A.D.4.) the unconstrained threshold beta β_U by its Long-only counterpart β_L as in (23).

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