Smart Beta: Managing Diversification of Minimum Variance Portfolios

Jean-Charles Richard* and Thierry Roncalli*†

*Lyxor Asset Management[1], France

†University of Évry, France

Risk Based and Factor Investing Conference

Imperial College, London, UK

November 5, 2015

[1] The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Lyxor Asset Management.

Summary

Main result

The difference in ex-post performance is mainly explained by the ex-ante level of volatility reduction targeted by smart beta portfolios. The choice of the diversification metric is marginal.

$\Rightarrow$ Two consequences:

1. Management report
2. Performance attribution

Risk-based portfolios

Main objective

The EW portfolio

$$x_i = x_j$$

Weights are equal.

The ERC portfolio

$$RC_i = RC_j$$

Risk contributions are equal.

The GMV portfolio

$$\min \frac{1}{2} x^\top \Sigma x$$

Minimize the volatility.

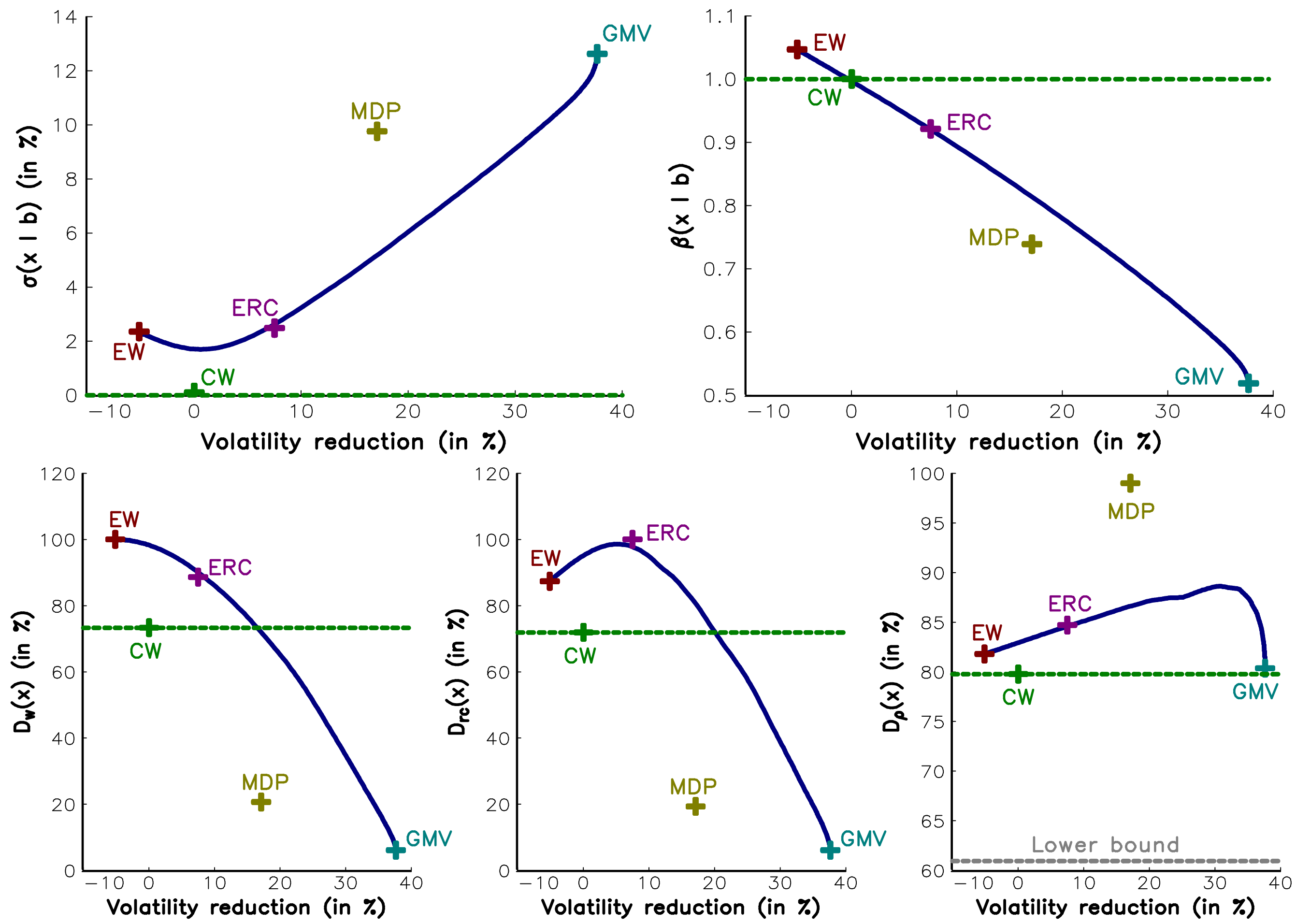
The MDP portfolio

$$\max \frac{x^\top \sigma}{\sqrt{x^\top \Sigma x}}$$

Maximize the diversification ratio.

Risk-based portfolios

GMV optimization program



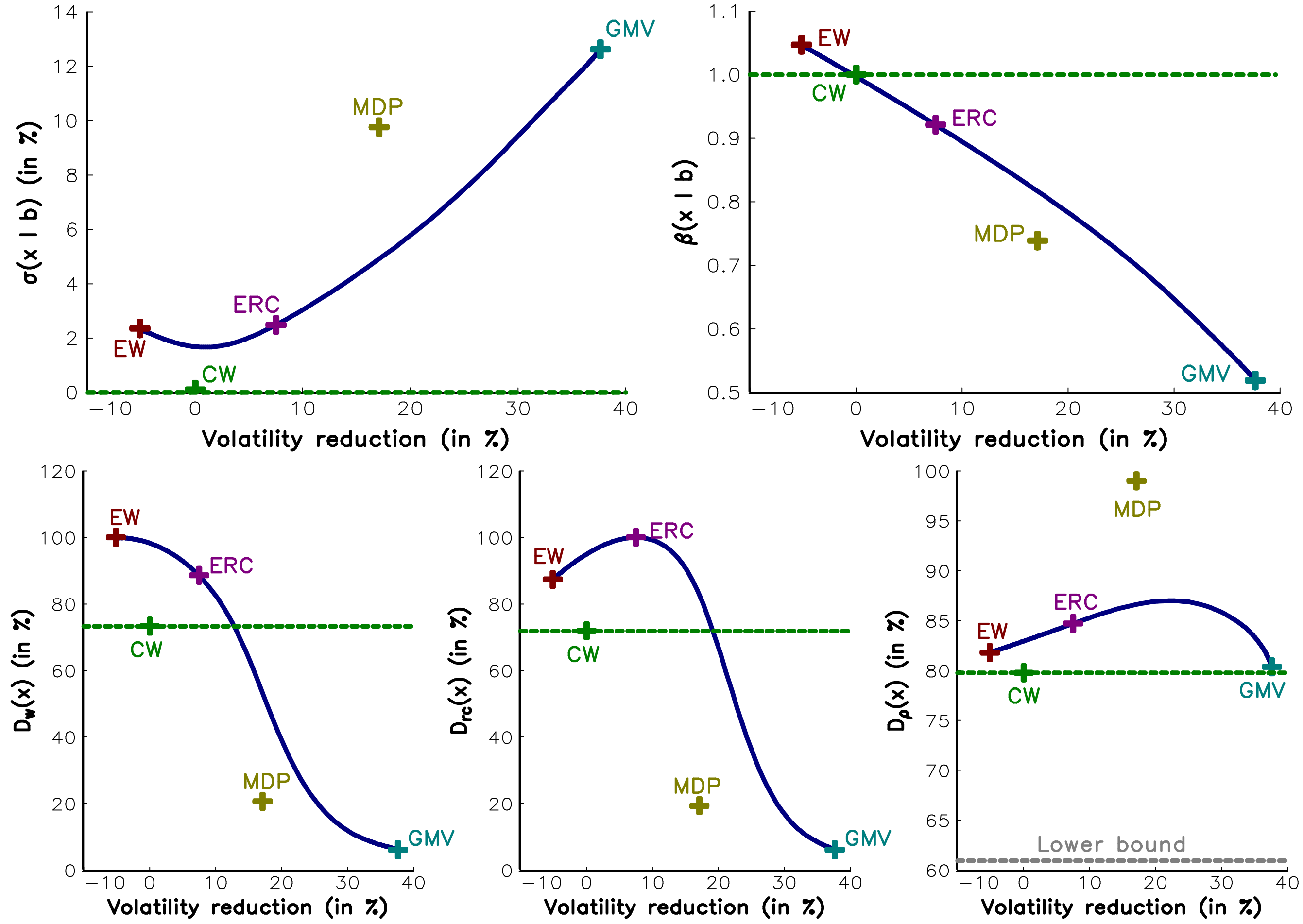
$$\arg\min \frac{1}{2} x^\top \Sigma x$$

$$\text{u.c.} \begin{cases} \mathbf{1}^\top x = 1 \\ \sum_{i=1}^n x_i^2 \leq c_1 \\ x \geq \mathbf{0} \end{cases}$$

*Euro Stoxx 50 Index — One-year empirical covariance matrix — February 2013.

Risk-based portfolios

ERC optimization program



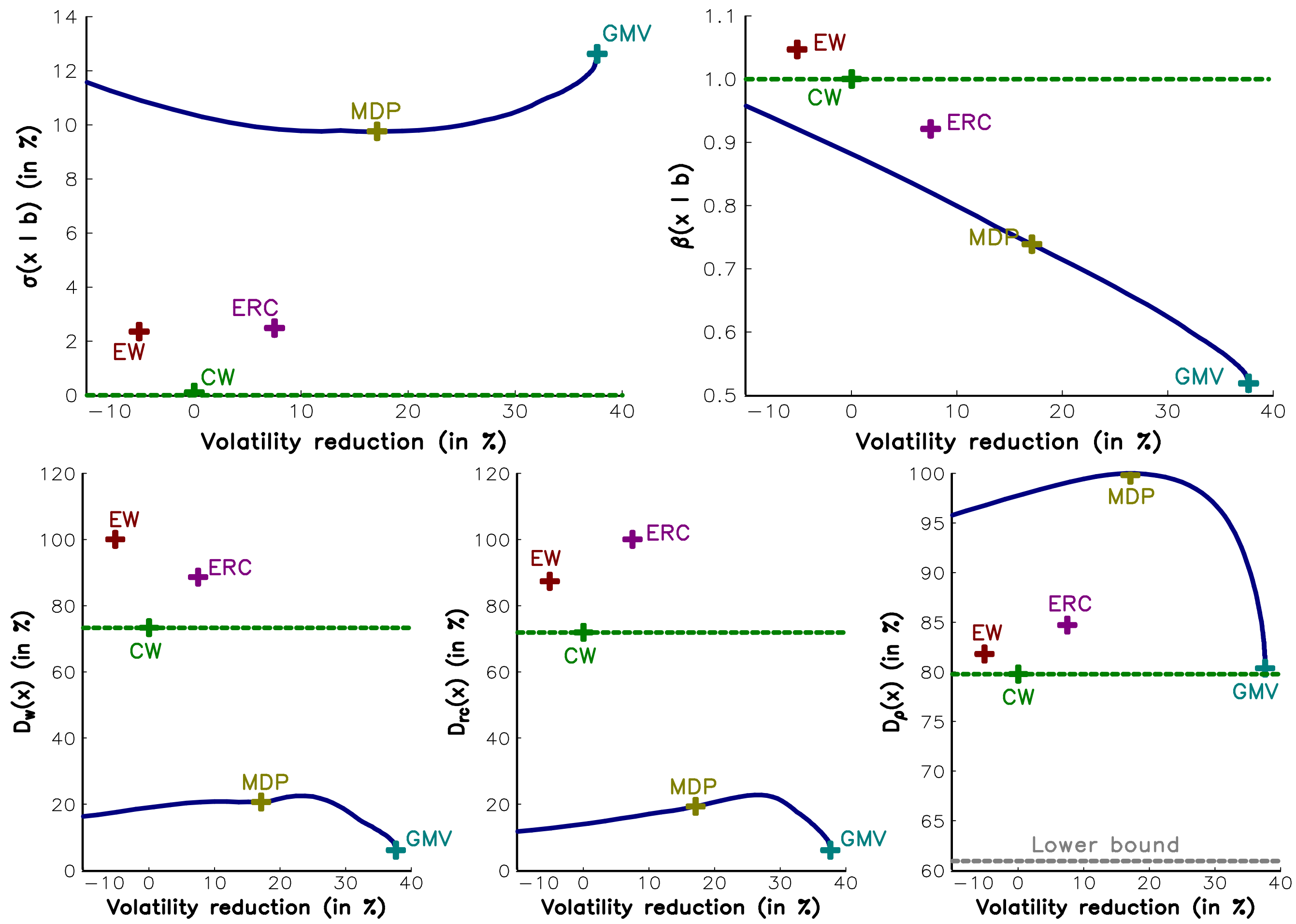
$$\arg\min \frac{1}{2} x^\top \Sigma x$$

$$\text{u.c.} \left\{ \begin{array}{l} \mathbf{1}^\top x = 1 \\ \cancel{\sum_{i=1}^n x_i^2 \leq c_1} \\ \sum_{i=1}^n \ln x_i \geq c_2 \\ x \geq \mathbf{0} \end{array} \right.$$

*Euro Stoxx 50 Index — One-year empirical covariance matrix — February 2013.

Risk-based portfolios

MDP optimization program



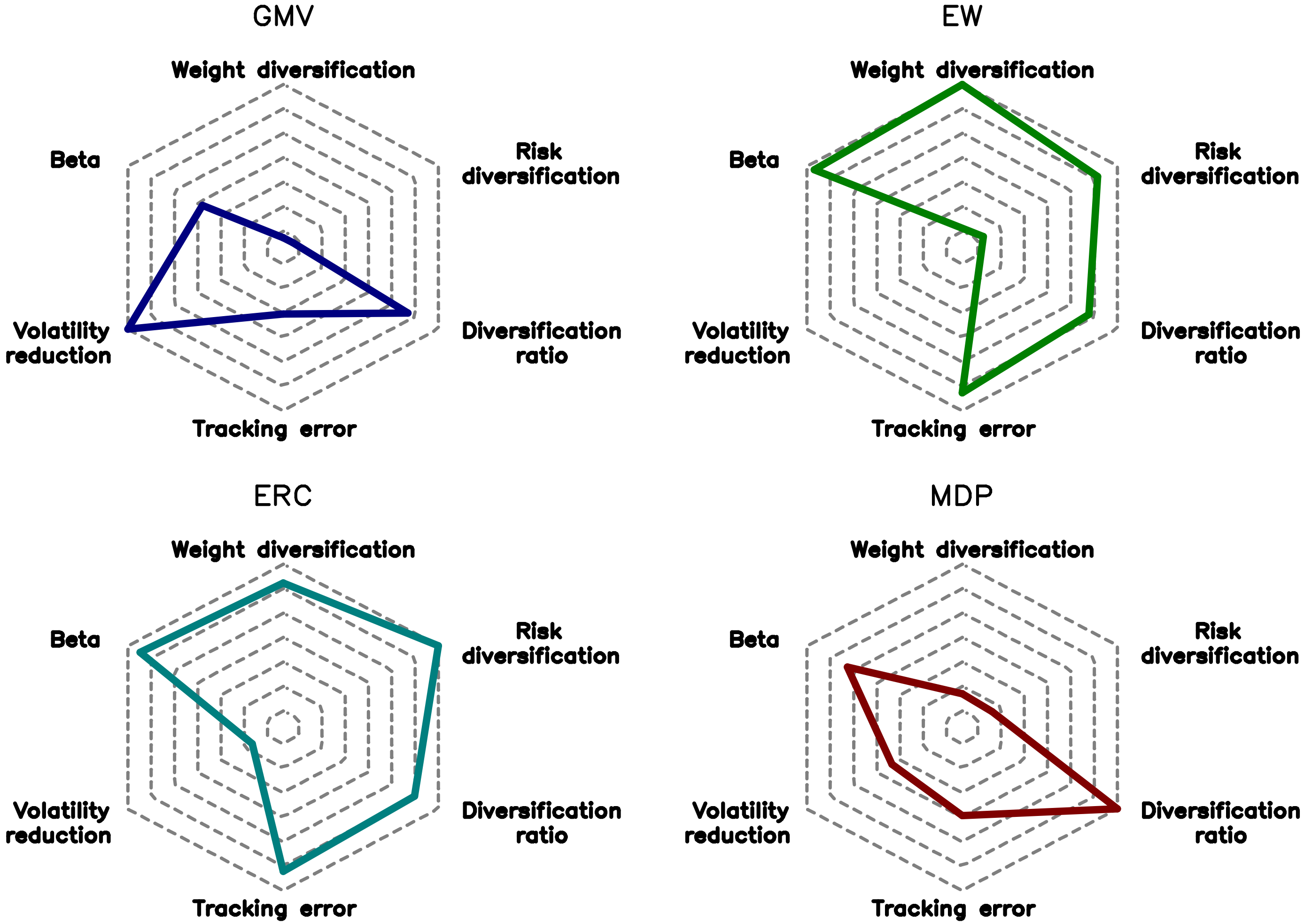
$$\arg\min \frac{1}{2} x^\top \Sigma x$$

$$\text{u.c.} \begin{cases} \mathbf{1}^\top x = 1 \\ \cancel{\sum_{i=1}^n x_i^2 \leq c_1} \\ \sum_{i=1}^n x_i \sigma_i \geq c_3 \\ x \geq \mathbf{0} \end{cases}$$

*Euro Stoxx 50 Index — One-year empirical covariance matrix — February 2013.

Diversification profile of risk-based portfolios

Figure: The case of Euro Stoxx 50 Index in February 2013



Mixing the constraints

Each risk-based portfolio is a minimum variance portfolio under a specific constraint:

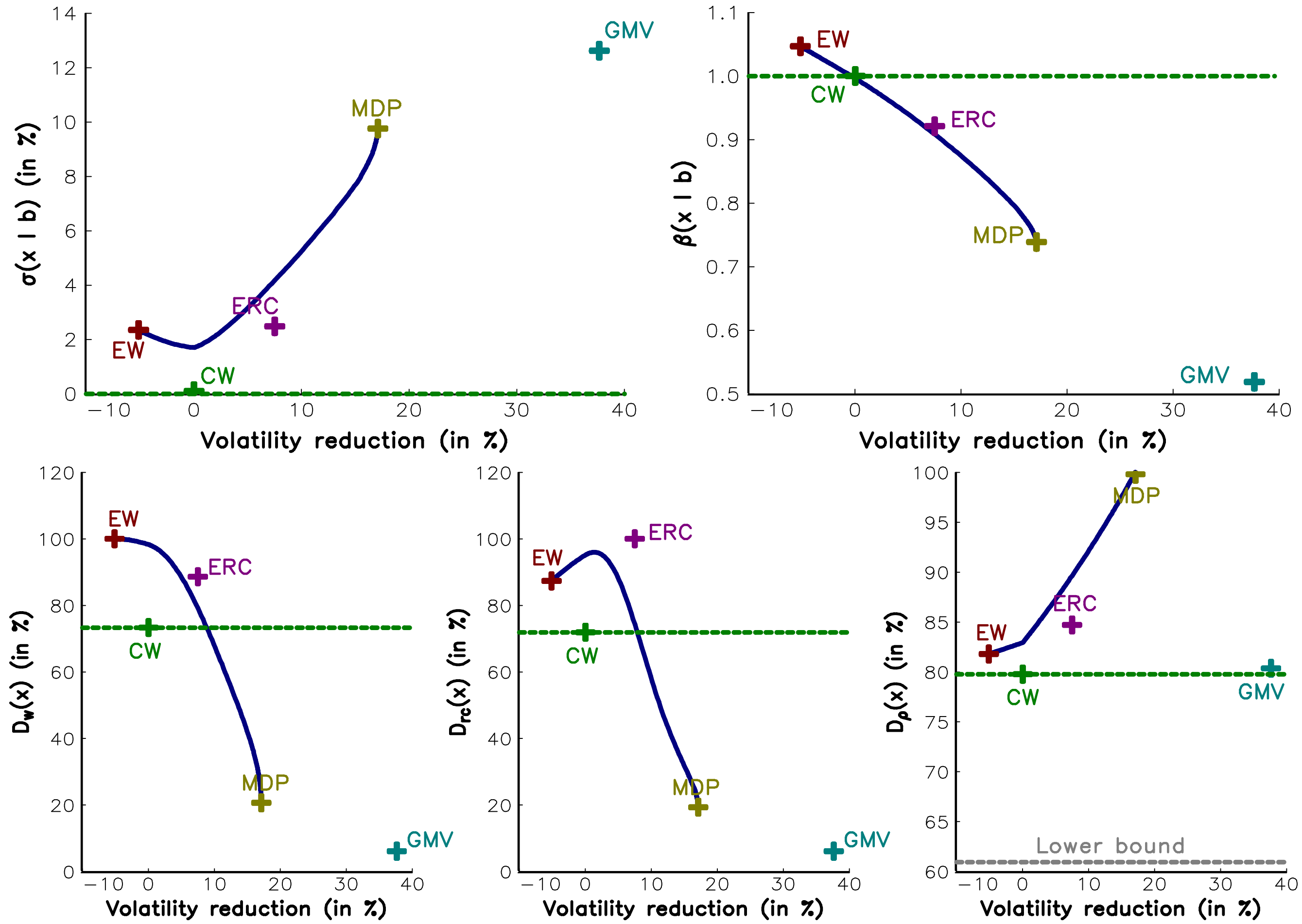
$$\begin{array}{ll} \mathbf{1}^\top x = 1 & \text{(GMV)} \\ \sum_{i=1}^n x_i^2 \leq c_1 & \text{(EW)} \\ \sum_{i=1}^n \ln x_i \geq c_2 & \text{(ERC)} \\ \sum_{i=1}^n x_i \sigma_i \geq c_3 & \text{(MDP)} \end{array}$$

Mixing the constraints

We can combine these different constraints to obtain better diversified risk-based portfolios. The first and fourth constraints allow the GMV portfolio and the MDP respectively to be obtained. The second and third constraints manage the diversification in terms of weights and risk contributions.

Mixing the constraints

An example (EW – MDP)



$$\arg\min \frac{1}{2} x^\top \Sigma x$$

$$\text{u.c.} \left\{ \begin{array}{l} \cancel{\mathbf{1}^\top x = 1} \\ \sum_{i=1}^n x_i^2 \leq c_1 \\ \sum_{i=1}^n x_i \sigma_i \geq c_3 \\ x \geq \mathbf{0} \end{array} \right.$$

*Euro Stoxx 50 Index — One-year empirical covariance matrix — February 2013.

A unified optimization framework

We can write the constrained problem using Lagrange multipliers:

$$\begin{aligned} x^{\star} &= \arg\min \frac{1}{2} x^{\top} \Sigma x - \\ & \lambda_{\mathrm{gmv}} \left(\sum_{i=1}^{n} x_i \right) + \lambda_{\mathrm{h}} \left(\sum_{i=1}^{n} x_i^2 \right) - \\ & \lambda_{\mathrm{erc}} \left(\sum_{i=1}^{n} \ln x_i \right) - \lambda_{\mathrm{mdp}} \left(\sum_{i=1}^{n} x_i \sigma_i \right) \\ \text{u.c.} \quad & x \geq \mathbf{0} \end{aligned} \tag{1}$$

Remark

The previous framework can be extended by replacing the variance minimization problem by the tracking error minimization problem. In this case, Problem (1) must include a new penalty function which is equal to: $-\lambda_{\mathrm{te}} \left(\sum_{i=1}^{n} x_i \left(\Sigma x_{\mathrm{cw}} \right)_i \right) = -\lambda_{\mathrm{te}} \beta \left(x \mid x_{\mathrm{cw}} \right) \sigma^2 \left(x_{\mathrm{cw}} \right)$

A unified optimization framework

The first-order condition is:

$$\frac{\partial \mathcal{L}(x)}{\partial x_i} = (\Sigma x)_i - \lambda_{\mathrm{gmv}} + 2\lambda_{\mathrm{h}} x_i - \frac{\lambda_{\mathrm{erc}}}{x_i} - \lambda_{\mathrm{mdp}} \sigma_i - \lambda_{\mathrm{te}} (\Sigma x_{\mathrm{cw}})_i = 0$$

The solution is the positive root of the second degree (convex) equation:

$$x_i^2 \left(\sigma_i^2 + 2\lambda_{\mathrm{h}}\right) + x_i \left(\sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j - \lambda_{\mathrm{gmv}} - \lambda_{\mathrm{mdp}} \sigma_i - \lambda_{\mathrm{te}} (\Sigma x_{\mathrm{cw}})_i\right) - \lambda_{\mathrm{erc}} = 0$$

We finally obtain the following CCD numerical solution:

$$\begin{aligned} x_i^\star &= \frac{\lambda_{\mathrm{gmv}} + \lambda_{\mathrm{mdp}} \sigma_i + \lambda_{\mathrm{te}} (\Sigma x_{\mathrm{cw}})_i - \sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j}{2\left(\sigma_i^2 + 2\lambda_{\mathrm{h}}\right)} \\ &+ \frac{\sqrt{\left(\sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j - \lambda_{\mathrm{gmv}} - \lambda_{\mathrm{mdp}} \sigma_i - \lambda_{\mathrm{te}} (\Sigma x_{\mathrm{cw}})_i\right)^2 + 4\left(\sigma_i^2 + 2\lambda_{\mathrm{h}}\right) \lambda_{\mathrm{erc}}}}{2\left(\sigma_i^2 + 2\lambda_{\mathrm{h}}\right)} \end{aligned}$$

A unified optimization framework

It is not possible to match all the diversification constraints

- Only a subset of Lagrange multipliers is interesting from a mathematical (and financial) point of view

This is equivalent to imposing the following constrained structure:

$$
\begin{aligned}
x^\star &= \arg\min \frac{1}{2} x^\top \Sigma x \\
& \text{u.c.} \quad \left\{ \begin{array}{l} \mathcal{D}(x;\gamma) \geq c_1 \\ \mathcal{B}(x;\delta) = c_2 \\ x \geq \mathbf{0} \end{array} \right.
\end{aligned}
$$

where $\mathcal{D}(x;\gamma)$ and $\mathcal{B}(x;\delta)$ are the diversification and budget constraints:

$$
\begin{aligned}
\mathcal{D}(x;\gamma) &= \gamma \sum_{i=1}^n \ln x_i - (1-\gamma) \sum_{i=1}^n x_i^2 \quad \text{(ERC / EW)} \\
\mathcal{B}(x;\delta) &= \delta \sum_{i=1}^n x_i + (1-\delta) \sum_{i=1}^n x_i \sigma_i \quad \text{(GMV / MDP)}
\end{aligned}
$$

Families of well-defined risk-based portfolios

The parameter $\gamma \in [0,1]$ controls the trade-off between weight and risk diversification whereas the parameter $\delta \in [0,1]$ controls the budget allocation.

We can then restrict (c_1, c_2) by considering this optimization problem:

$$\begin{aligned} x^\star(\lambda, \gamma, \delta) &= \arg\min \frac{1}{2} x^\top \Sigma x - \lambda \mathscr{D}(x;\gamma) + (\lambda - 1)\mathscr{B}(x;\delta) \\ \text{u.c.} \quad & x \geq \mathbf{0} \end{aligned} \quad (2)$$

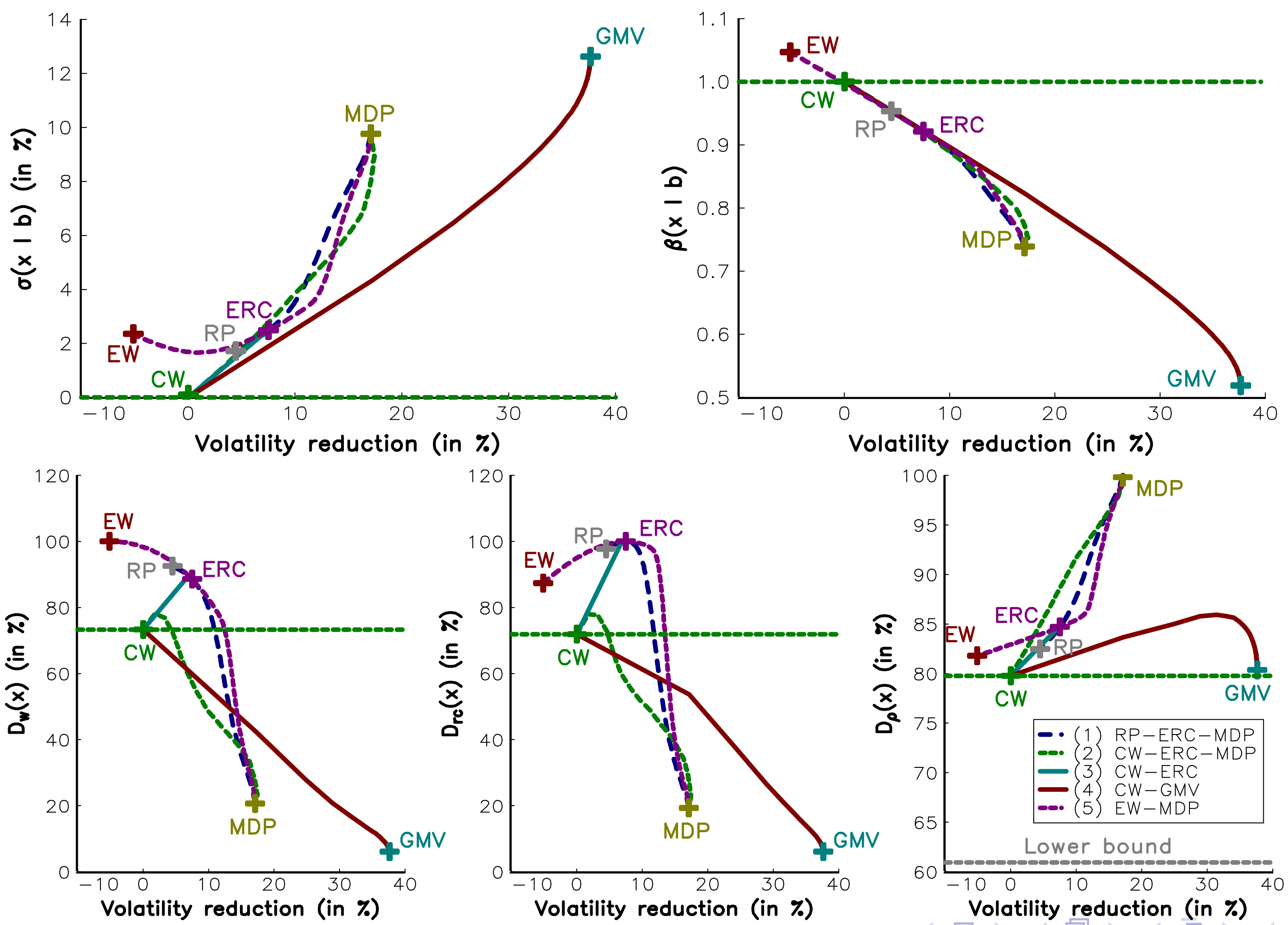
where $\lambda \geq 0$ controls the impact on the diversification.

Parameters	GMV	EW	ERC	MDP	RP	BP
λ	0	$+\infty$	1	0	$+\infty$	$+\infty$
γ		0/1	1		1	1
δ	1	1		0	1	0

$\Rightarrow$ Extension to the tracking-error volatility ($\Rightarrow \mathscr{B}(x;\delta)$).

Examples

Figure: New families of smart beta portfolios



No free lunch in smart beta

Rule 1

There is no free lunch in smart beta. In particular, it is not possible to target a high volatility reduction, to be highly diversified and to take low beta risk.

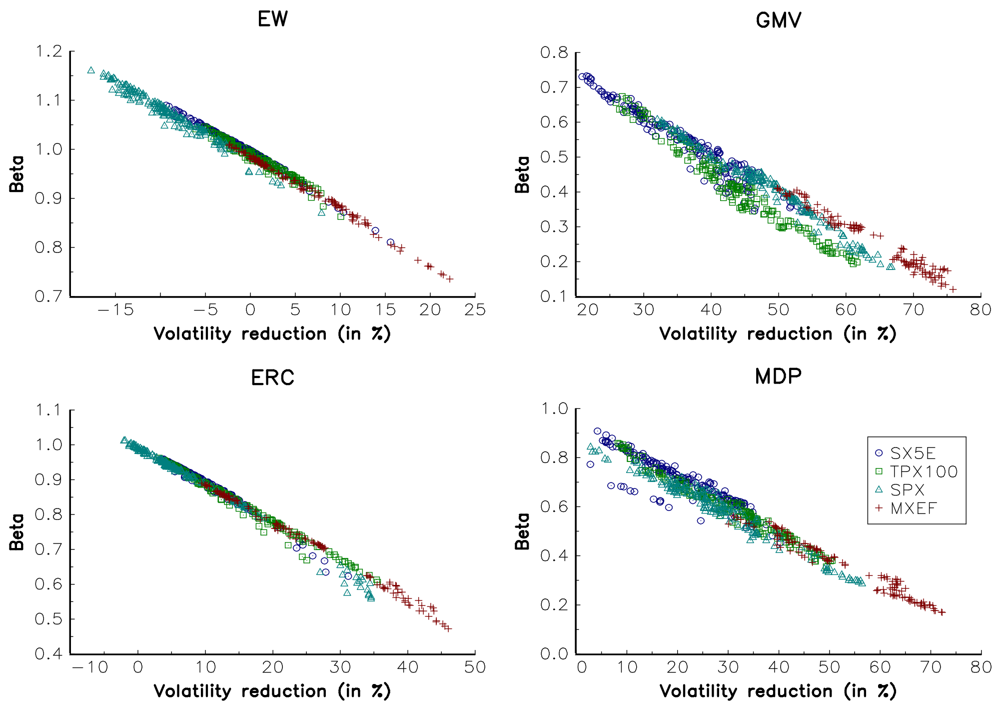


Figure: Relationship between the volatility reduction and the beta

Volatility reduction

Rule 2

The smart beta portfolios have a time-varying objective of volatility reduction and tracking error.

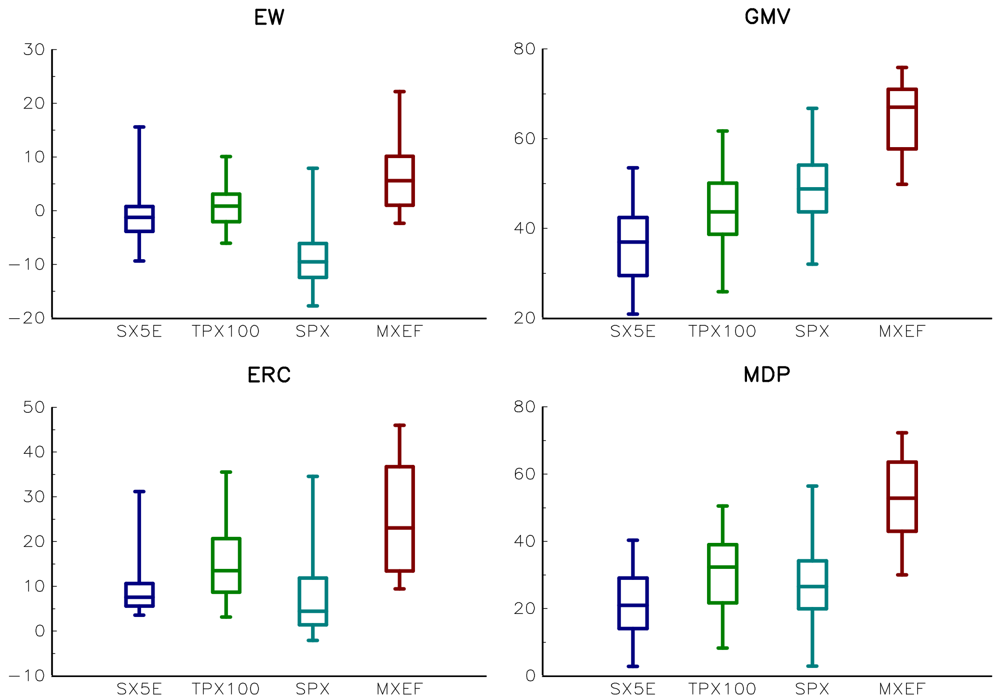


Figure: Boxplot of the volatility reduction (in %)

Ex-ante volatility reduction explains ex-post behavior

Rule 3

When we impose the same objective of volatility reduction $\eta^\star$, smart beta portfolios become comparable.

Table: Average correlation between risk-based portfolios (in %)

Index	$\eta^\star$	VR	TE	β	D_w	D_{rc}	D_ρ	R_t
SX5E	5%	100.0	99.2	100.0	99.3	99.5	99.8	100.0
	10%	100.0	92.1	99.5	86.7	71.6	98.9	99.8
	15%	100.0	91.5	97.4	88.6	76.4	97.2	99.2
TPX100	5%	100.0	99.8	100.0	99.7	99.8	99.9	100.0
	10%	100.0	88.3	98.9	89.1	65.0	98.2	100.0
	15%	100.0	91.5	97.6	92.7	78.4	97.5	99.9
SPX	5%	100.0	96.8	99.8	86.4	63.6	98.2	99.8
	10%	100.0	86.9	97.1	88.4	69.7	93.4	99.0
	15%	100.0	85.6	90.8	88.9	77.6	88.4	97.6
MXEF	5%	100.0	100.0	100.0	99.9	100.0	100.0	100.0
	10%	100.0	100.0	100.0	98.2	99.5	99.8	100.0
	15%	100.0	99.9	100.0	96.1	95.0	99.5	100.0
Average		100.0	94.3	98.4	92.8	83.0	97.6	99.6

Relationship between volatility reduction and excess return

Rule 4

The performance of smart beta portfolios depends on the market risk premium.

Figure: Jul. 2007-Feb. 2009

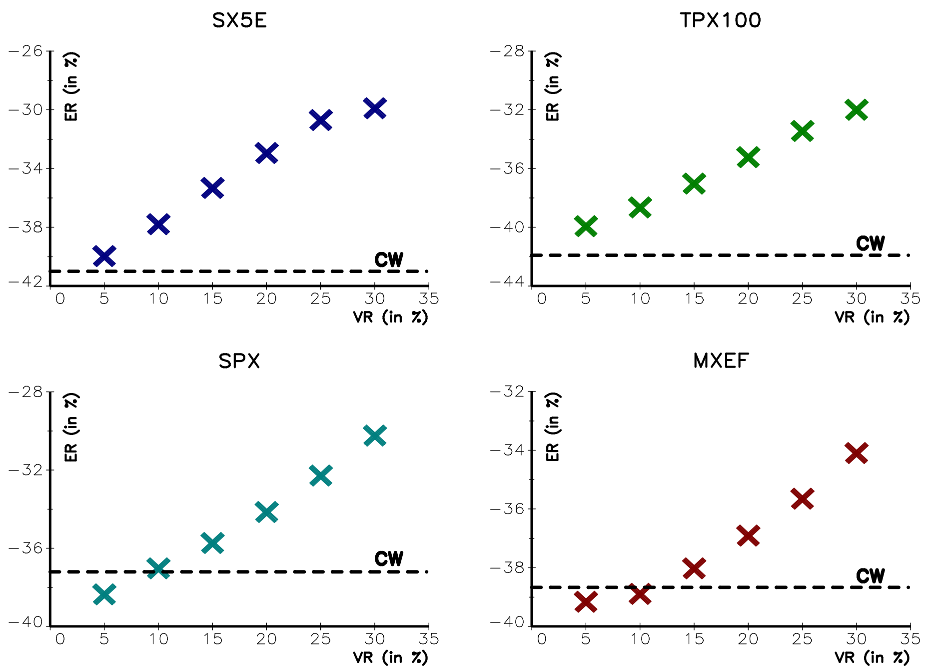
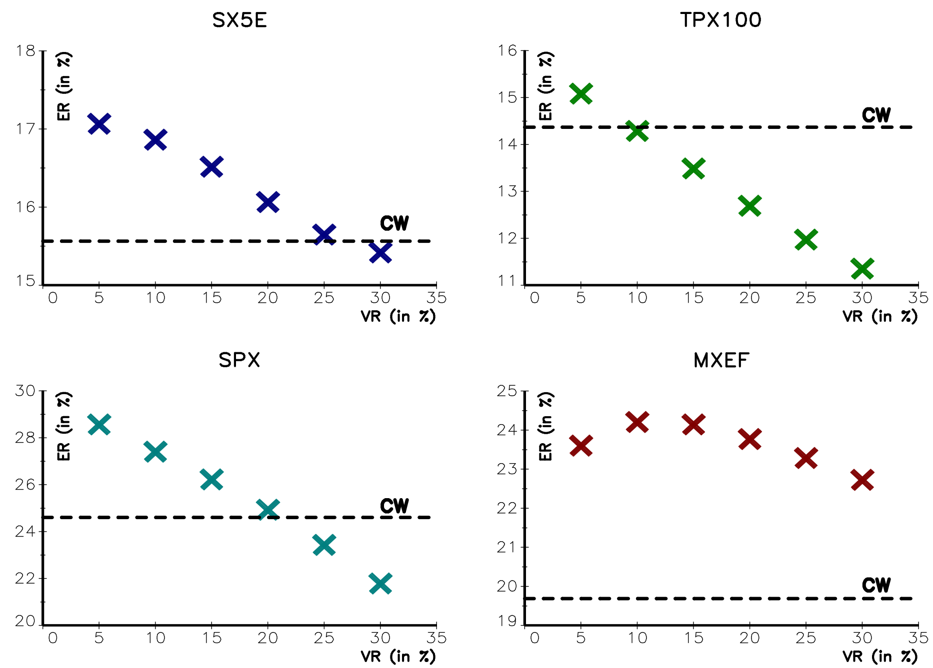


Figure: Mar. 2009-Dec. 2013



Managing the trade-off between volatility reduction and diversification

The previous rules can be used to build dynamic smart beta strategies.

When risk is perceived as high/low, we expect a lower/higher risk premium:

1. High level of volatility reduction;
2. High level of risk diversification.

We link the parameter λ in Problem (2) to the **market sentiment**, which is approximated by the cross-section (CS) volatility:

$$\lambda = 1 - \phi \frac{\sigma_t^{\mathrm{cs}} - \sigma_t^-}{\sigma_t^+ - \sigma_t^-}$$

and we impose that $\gamma = 1$ (ERC) and $\delta = 1$ (GMV).

Empirical results

- **Risk-off**: High $\sigma_t^{cs} \Rightarrow \lambda = 0$ (GMV) / **Risk-on**: Low $\sigma_t^{cs} \Rightarrow \lambda = 1$ (ERC).
- $\mathrm{D}_{\#1}$ corresponds to the case $\lambda \in [0,1]$.
- $\mathrm{D}_{\#2}$ corresponds to the case $\lambda \in [0.15,1]$.

Table: Comparing GMV, ERC and dynamic smart beta strategies (2001-2014)

	CW	GMV	ERC	$\mathrm{D}_{\#1}$	$\mathrm{D}_{\#2}$	CW	GMV	ERC	$\mathrm{D}_{\#1}$	$\mathrm{D}_{\#2}$
			SX5E					TPX100		
$\mu(x)$ (in %)	0.6	3.8	3.4	**5.1**	**4.7**	0.4	6.3	3.3	**3.6**	**3.2**
$\sigma(x)$ (in %)	24.5	19.1	23.1	21.3	22.4	24.4	16.3	21.3	18.9	19.8
$\mathrm{SR}(x)$	−0.1	0.1	0.1	0.1	0.1	0.0	0.4	0.1	0.2	0.1
$DD(x)$ (in %)	−59.6	−52.4	−54.4	−50.7	−51.5	−62.8	−49.4	−57.4	−51.1	−54.2
$\tau(x)$	0.2	3.4	0.8	3.0	1.9	0.3	3.8	1.0	2.9	1.8
			SPX					MXEF		
$\mu(x)$ (in %)	5.0	8.3	9.9	**11.5**	**10.5**	8.0	12.0	10.8	**14.3**	**12.6**
$\sigma(x)$ (in %)	20.1	12.2	19.2	16.2	18.2	21.6	9.4	16.3	13.0	14.3
$\mathrm{SR}(x)$	0.2	0.5	0.4	0.6	0.5	0.3	1.1	0.6	1.0	0.8
$DD(x)$ (in %)	−55.3	−33.3	−55.9	−44.7	−52.5	−65.1	−29.9	−53.8	−34.9	−44.9
$\tau(x)$	0.1	5.9	1.0	3.5	1.6	0.5	5.6	1.6	4.2	2.8