A New Benchmark for Dynamic Mean-Variance Portfolio Allocations*

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Abstract

We propose a new methodology to implement unconditionally optimal dynamic mean-variance portfolios. We model portfolio allocations using an auto-regressive process in which the shock to the portfolio allocation is the gradient of the investor's realized certainty equivalent with respect to the allocation. Our methodology can accommodate transaction costs, short-selling and leverage constraints, and a large number of assets. In out-of-sample tests using equity portfolios, long-short factors, government bonds, and commodities, we find that its risk-adjusted performance, net of transaction costs, is on average more than double that of other benchmark allocations.

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1 Introduction

Dynamic mean-variance portfolio choice plays a crucial role in financial economics and asset management. In a first step, portfolio choice involves deriving an investor's optimal allocation accounting for his long-term investment objective, the cost of intermediate trading, his allocation constraints, etc. Then, implementing this allocation further requires estimating parameters such as expected returns.

However, deriving and implementing dynamic mean-variance portfolios is notoriously difficult. Even in the case of a myopic—one-period—mean-variance investor with no frictions, many methodologies proposed in the literature do not outperform a naive equal-weighted allocation out-of-sample, see DeMiguel, Garlappi, and Uppal (2009b). This problem is even more severe in the case of an investor with a long-term investment objective and facing frictions. Consequently, Cochrane (2014) points out that:

> *"dynamic incomplete-market portfolio theory is widely ignored in practice, though it has been around for half a century. Even highly sophisticated hedge funds typically form portfolios with one-period mean-variance optimizers—despite the fact that mean-variance optimization for a long-run investor assumes i.i.d. returns, while the funds' strategies are based on complex models of time-varying expected returns, variances, and correlations."*

In this paper, we consider the problem of an investor who maximizes a long-run mean-variance objective while exploiting predictability in expected returns, variances, and correlations. We have in mind a mutual fund or a hedge fund using different signals about assets' risk/return profiles, and whose performance is evaluated by its unconditional (i.e., long-run) mean-variance tradeoff.[1] The optimal dynamic allocation in this context is known

[1] The literature on intertemporal portfolio choice often considers investors with preference over their wealth at a predetermined investment horizon. For example, Basak and Chabakauri (2010) solve for the time-consistent intertemporal portfolio solution of an investor with mean-variance preference over his terminal wealth. However, the investment horizon for some investors, such as mutual funds or hedge funds, is difficult to determine. We focus on investors with preference over the unconditional risk-adjusted performance of one-period portfolio returns instead of multi-periodic returns over a fixed investment horizon.

from Ferson and Siegel (2001) (FS).

Although the FS allocation differs from the standard one-period mean-variance allocation of Markowitz (1952), it too faces several implementation challenges. First, it requires a model of the dynamics of expected returns, variances, and correlations. While measuring these moments at any point in time is difficult, especially when the number of assets is large, modeling their *dynamics* is harder still. Second, deriving a closed-form solution to this problem becomes unwieldy once transaction costs and allocation constraints such as no short-selling are taken into account.[2] This paper provides a methodology for implementing unconditionally optimal dynamic mean-variance portfolio allocations with possibly a large number of assets, transaction costs, and allocation constraints.

1.1 Methodology overview

The large literature on implementation techniques for mean-variance portfolios, which we survey in a following section, can be organized around two key modeling decisions. The traditional approach derives analytically the portfolio allocations as a function of expected returns and the covariance matrix, and then substitute estimates of these *return moments* to obtain allocations. The first decision concerns the information used to measure these return moments.

One approach is to estimate the return moments using historical returns and plug these sample estimates into the optimal mean-variance weight function (see Brandt, 2010, for a review). DeMiguel et al. (2009b) (hereafter DGU) offer a sobering account of the effectiveness of these techniques which, by and large, fail to outperform an equal-weighted allocation out-of-sample.[3]

Another approach is to build models to predict each return moment using a broader information set such as firm characteristics and macroeconomic variables (see, for example,

[2] Assuming constant variances and correlations, Gârleanu and Pedersen (2013) derive the optimal allocation with predictable expected returns and quadratic transaction costs.

[3] However, there is evidence that estimates of the conditional covariance matrix can be used to improve portfolio performance, see for example Fleming, Kirby, and Ostdiek (2001, 2003).

Ferson and Siegel, 2009; Marquering and Verbeek, 2004). A major impediment to this approach is the proliferation of parameters to estimate as the number of assets, and hence the number of return moments, grows.

Whether one uses past returns or other variables to model return moments, making sure that good return moment predictions translate into quality *portfolio allocations* is challenging. Consider for example the one-period Markowitz portfolio allocation that is proportional to $\Sigma^{-1}\mu$ where Σ is the covariance matrix of asset returns and μ is the vector of expected returns in excess of the risk-free rate. As the covariance matrix is inverted and then multiplied by the expected returns, the compounding of measurement errors made on each return moment is complex, and it is not reasonable to assume that these errors cancel out. This problem remains even for state-of-the-art models for predicting expected returns (e.g., Gu, Kelly, and Xiu, 2019) or the covariance matrix (see, for example, Bollerslev, Hood, Huss, and Pedersen, 2018; Engle, Ledoit, and Wolf, 2019; Lucas, Schwaab, and Zhang, 2014).

The second key modeling decision that distinguishes techniques is whether one uses the two-step process described above or instead directly models portfolio allocations as a function of instruments (i.e., firm characteristics and macroeconomic variables). Brandt, Santa-Clara, and Valkanov (2009) relate instruments to an investor's optimal allocation to a large cross-section of stocks. They avoid the intermediary step in which return moment estimates are translated into portfolio allocations, and directly model the latter as a function of instruments.

The drawback of this approach is that it requires identifying common instruments that correlate with optimal allocations across all assets, and second a large cross-section of assets to estimate these correlations. Therefore, this approach can hardly be applied when an investor allocates to a few risk factors or broad asset classes. Furthermore, the handling of transaction costs and allocation constraints has to be done by constraining the numerical optimization. This brute-force approach is impractical given that all investors face trading costs and most have allocation constraints (e.g., mutual funds are usually prohibited from short-selling).

This paper introduces a new approach to modeling portfolio allocations when either there are no common instruments, the cross-section of assets is small, or both. The methodology can additionally accommodate transaction costs and allocation constraints. We model portfolio allocations using a mean-reverting dynamic. Each period, the optimal portfolio allocations are a weighted average of three components: the long-run allocations which capture the investor's strategic asset allocation, the portfolio allocations from the previous period, and a shock to portfolio allocations. The second and third components capture the investor's temporary deviations from his long-run allocation (i.e., his tactical asset allocation). The weights used to compute the weighted average of the three components are parameters estimated using past returns by maximizing the investor's unconditional mean-variance criterion. The estimated parameters of the dynamic indicate the extent to which deviating from the long-run asset allocation, and hence engaging in tactical asset allocation, is beneficial.

The main difficulty in using such a time-series modeling approach for portfolio allocations is to find appropriate shocks to allocations. The key contribution of our methodology is our characterization of such shocks; we use the gradient of the unconditional mean-variance criterion with respect to the current period allocation as the optimal shock to portfolio allocation next period.

The intuition for this choice of allocation shock is the following. At time t, the gradient indicates the direction in which to change the allocation to improve the unconditional mean-variance criterion. Unfortunately, computing the gradient requires the asset returns realized at time t and, therefore, cannot be used in real-time for portfolio allocation at the beginning of period t as they are not known yet. However, if expected returns and covariances are time-persistent, past gradients can be used to optimize the current allocation. In our methodology, the estimated weight parameters for the auto-regressive (i.e., the previous period allocation) and shock components accomplish this task. We refer to this approach as auto-regressive portfolio allocations (ARPA).

Our methodology offers several advantages. The gradient can easily be computed for mean-variance preferences, is a non-linear function of past returns, complements firm characteristics used in other studies, and can be used when the cross-section of assets is small or other instruments are not easily identifiable. Also, our methodology accommodates allocation constraints such as no short-selling and allocations summing to one. Finally, our methodology implicitly accounts for transaction costs because the parameters are estimated on realized returns net of transaction costs. The extent to which the estimated parameters allow for temporary deviations from long-run allocations reflects how the methodology optimally weighs the performance benefits versus the increase in portfolio turnover.

1.2 Findings

In empirical tests of our approach, we use different sets of assets that include equity factors, equity portfolios, government bonds, and commodities. For each week, we estimate the parameters of the ARPA model using past returns and compute the realized (i.e., out-of-sample) return on the optimal allocation. We also compute the realized performance of two other allocations to benchmark our results. The first is the equal-weighted portfolio (EW), which we use to measure the benefit of optimized allocations based on our approach versus a naive allocation. The second is the sample-based mean-variance portfolio (SMV) based on estimates of the expected returns and covariance matrix using all past returns. While we confirm DGU's findings that the SMV does not consistently outperform the EW portfolio, we use the SMV portfolio to assess the benefits of dynamic rebalancing implied by the ARPA method in the few cases where SMV outperforms EW.

On average across all sets of assets and risk aversion levels, the ARPA portfolio more than doubles the realized mean-variance criterion of the EW portfolio. This average increase is significant, with a t-ratio larger than four. The ARPA portfolio also outperforms the SMV portfolio in cases where the latter outperforms the EW portfolio, suggesting that dynamic rebalancing adds values. However, there are important differences across sets of assets and risk aversion levels. To gain a better understanding of the benefits of the ARPA methodology, we run panel regressions to explain the weekly performance contributions of the ARPA portfolio relative to those of the EW portfolio.

We empirically investigate the role of several determinants of outperformance. First, DGU analytically show that there are three conditions under which the SMV portfolio is expected to outperform an equal-weighted allocation: *(i)* a long sample to estimate parameters, *(ii)* a small number of assets, and *(iii)* a large difference in expected squared Sharpe ratios between the two portfolios. They demonstrate that the minimum requirement on the sample size implied by these three conditions is so strict that it makes the SMV portfolio infeasible in a realistic setting.

The outperformance of the ARPA portfolio is positively related to the length of the estimation sample. The ARPA portfolio, just as the SMV, performs better when a long estimation sample is available. Next, we find a positive coefficient for the number of assets, which is the opposite of what condition *(ii)* for the SMV portfolio implies. Therefore, the ARPA methodology is not afflicted by the curse of dimensionality.

Although we find that the outperformance is not related to the expected difference in Sharpe ratios, it is positively associated with the mean absolute difference in portfolio allocations. This result is intuitive: if the optimal allocation is close to the equal-weighted allocation and we face estimation risk, then adopting the equal-weighted allocation is preferable. On the other hand, when the ARPA portfolio differs from the EW portfolio, it generates significant outperformance.

We then examine the dynamic allocations over time. We find that the generated outperformance comes from persistent leverage, especially for low risk aversion levels, and small but persistent short positions. These results are important for unconstrained investors, such as hedge funds, who follow leveraged dynamic long-short strategies.

Next, we examine the performances of the dynamic allocations for investors constrained to be fully invested in risky assets and to only take long positions. This case is typical of mutual funds. We find that, for the sets of assets for which ARPA delivers outperformance in the unconstrained case, it does so too in the constrained case for low risk aversion levels. However, the outperformance is smaller than in the unconstrained case. Across all sets of assets and risk aversion levels, the increase in realized mean-variance criterion is 25% relative to the EW portfolio. This average increase is significant, with a t-ratio of 2.71.

Our results are important because imposing allocation constraints usually improves the performance of optimized allocations, (see, for example, Jagannathan and Ma, 2003; DeMiguel, Garlappi, Nogales, and Uppal, 2009a). In our case, we instead find better out-of-sample performance for unconstrained portfolios, suggesting that the ARPA model is less affected by estimation errors than other methodologies.

1.3 Related literature

Our paper contributes to the literature on parametric portfolio rules. Brandt (1999) and Ait-Sahalia and Brandt (2001) use a kernel estimator to relate instruments to optimal portfolio allocations. The non-parametric nature of their methodology renders difficult its application to a large number of assets, whereas our methodology can be applied in high-dimensional contexts.

Brandt et al. (2009) generalize this approach using a parametric function that relates instruments for a large number of assets to their portfolio allocations. This parametric portfolio rule has been successfully applied by Ghysels, Plazzi, and Valkanov (2016) to allocate across emerging market stocks, by Barroso and Santa-Clara (2015) to form currency portfolios, and Bredendiek, Ottonello, and Valkanov (2019) to form bond portfolios. More recently, DeMiguel, Utrera, Nogales, and Uppal (2019) combine this methodology with a LASSO penalty to select among a large number of instruments. Their approach handles both a large number of stocks and a large number of instruments, offering a genuinely big data approach to portfolio management. First, our methodology complements these studies because the allocation shock (i.e., the gradient of the realized mean-variance criterion) represents a new instrument based solely on past returns. But in addition, our methodology can be applied when either instruments are not shared across assets or the cross-section is not large enough to estimate the relation between instrument values and optimal allocations.

Brandt and Santa-Clara (2006) augment a set of assets by creating many managed portfolios (assets scaled by an instrument) to transform a complicated dynamic portfolio choice problem in a simpler static one. Our methodology is similar to theirs in the sense that both focus on a parametric rule for portfolio allocations that maximizes an unconditional mean-variance criterion and therefore offer a practical way of implementing the FS portfolio. However, we differ from their approach in that we do not use instruments to parameterize portfolio allocations, or said differently, we propose a new instrument for portfolio allocations based only on past returns. In addition, their focus is on the in-sample significance of different instruments whereas ours is a wide empirical investigation of the out-of-sample performance of our methodology accounting for transaction costs and allocation constraints.

The findings of DeMiguel et al. (2009b) have prompted several to propose new methods to implement the one-period optimal mean-variance portfolio. Kirby and Ostdiek (2012a) identify turnover as the main culprit for the poor performance of mean-variance portfolios and propose two low-turnover strategies. In our methodology, the estimated parameters explicitly weigh the tradeoff between higher performance and turnover through the impact of transaction costs on performance. Building on Kan and Zhou (2007), Tu and Zhou (2011) show how to combine solutions to the one-period Markowitz portfolio rule with the equal-weighted allocation. Our methodology allows incorporating allocation constraints and the impact of transaction costs. Anderson and Cheng (2016) provide a Bayesian-averaging approach to combine estimates of the expected returns and covariance matrix obtained from different lookback periods. Similarly, the parameters of our methodology determine the optimal weight to put on past observations. Kirby and Ostdiek (2012b) estimate expected returns and the covariance matrix using a weighted moving average calibrated to maximize an unconditional mean-variance criterion. Our methodology instead directly models portfolio allocations and sidesteps the estimation of return moments. Ao, Li, and Zheng (2019) propose a penalized regression-based approach to form optimal static mean-variance portfolios with a large number of assets. Whereas their approach allows for a risk constraint, our methodology also allows for allocation constraints such as no short-selling. Using the extension of the APT from Uppal, Zaffaroni, and Zviadadze (2020), Raponi, Uppal, and Zaffaroni (2020) form misspecification-robust mean-variance portfolios with a large number of assets. Our methodology can be applied to a small number of assets, and therefore complements the methodologies of Ao et al. (2019) and Raponi et al. (2020).

Finally, our approach is also related to auto-regressive score models used in risk management. Creal, Koopman, and Lucas (2011) model time-variations in the covariance matrix of returns using the gradient of the log-likelihood function. See also Lucas et al. (2014) for an application to modeling risk in euro area sovereign debt. In our portfolio management context, we instead use the gradient of the unconditional mean-variance criterion to model time-variations in allocations.

The paper proceeds as follows. Section 2 compares different mean-variance portfolios and presents our methodology. Section 3 presents our empirical results. Section 4 concludes.

2 Optimal mean-variance portfolio allocations

We consider an investor maximizing his unconditional mean-variance criterion (UMVC) as,

$$\max_{w_t} E[r_{p,t}] - \frac{\gamma}{2} Var\left(r_{p,t}\right), \tag{1}$$

where γ is his risk aversion, the N-by-one vector of portfolio allocations w_t can vary each period, $r_{p,t} = r_f + w_t' r_t$ is the portfolio return at time t, and r_t is the N-by-one vector of excess returns at time t. We refer to this objective as *unconditional* because we use the unconditional expected portfolio return, $E[r_{p,t}]$, and portfolio variance, $Var\left(r_{p,t}\right)$. In this section, we contrast different portfolio rules one can use to obtain w_t.

In the myopic approach, the investor implements each period the one-period optimal mean-variance portfolio as,

$$w_t^{Markowitz} = \frac{\Sigma_t^{-1}\mu_t}{\gamma}, \tag{2}$$

where Σ_t and μ_t are the covariance matrix and expected value of the vector r_t conditional on information at time $t-1$, with typical element $\Sigma_{t,i,j} = Cov_{t-1}(r_{t,i}, r_{t,j})$ and $\mu_{t,i} = E_{t-1}[r_{t,i}]$.

Implementing portfolio rule $w_t^{Markowitz}$ requires to plug estimates of Σ_t and μ_t into Equation (2). Implementation techniques for this approach mainly differ by their choice of estimators for Σ_t and μ_t, as in the case of shrinkage estimators for example. See DGU for a review of existing methods and their relative performance.

However, using portfolio rule (2) each period does not necessarily maximize the objective (1). The rule $w_t^{Markowitz}$ maximizes the conditional mean-variance criterion $E_{t-1}[r_{p,t}] - \frac{\gamma}{2} Var_{t-1}(r_{p,t})$ and therefore would maximize over T periods the sum of the conditional criteria as,

$$\max_{w_t} \frac{1}{T} \sum_{t=1}^{T} E_{t-1}[r_{p,t}] - \frac{\gamma}{2} Var_{t-1}(r_{p,t}). \tag{3}$$

In this equation, we have divided by T to highlight the difference with the *unconditional* mean-variance criterion (1). Since $E[E_{t-1}[r_{p,t}]] = E[r_{p,t}]$, the average conditional expected portfolio return, $\frac{1}{T}\sum_{t=1}^{T} E_{t-1}[r_{p,t}]$, converges to the unconditional expected portfolio return, $E[r_{p,t}]$, as T grows. However, the law of total variance, $Var(r_{p,t}) = E[Var_{t-1}(r_{p,t})] + Var(E_{t-1}[r_{p,t}])$, implies that the average conditional portfolio return variance, $\frac{1}{T}\sum_{t=1}^{T} Var_{t-1}(r_{p,t})$, does not converge to the unconditional variance, $Var(r_{p,t})$, unless conditional expected portfolio returns do not vary.

This issue is related to the problem of time-consistency in intertemporal portfolio choice. In that context, the investor expects at time t the variance of his wealth at the end of his investment horizon to be higher than the variance expected at a future date. Consequently, the investor has an incentive to deviate at a later date from the optimal investment policy he had chosen at time t. Basak and Chabakauri (2010) provide the time-consistent investment policy from which the investor has no incentive to deviate. In the context of the myopic Markowitz allocation, the problem stems from using a series of conditionally optimal allocations to maximize an unconditional criterion that differs from the expected value of the conditional criteria.

Ferson and Siegel (2001) derive the solution that maximizes (1) when the investor uses conditioning information as

$$w_t^{FS} = \frac{1}{\gamma\left(1-\zeta\right)}\left(\left(\mu_t\mu_t' + \Sigma_t\right)^{-1}\mu_t\right). \tag{4}$$

where $\zeta = E\left[\mu_t'\left(\mu_t\mu_t' + \Sigma_t\right)^{-1}\mu_t\right]$. The main difference between the Markowitz rule (2) and the FS rule comes from the presence of $\mu_t\mu_t'$ in the denominator of the rightmost term of Equation (4) that makes the optimal allocation a non-linear function of μ_t.

The allocations $w_t^{Markowitz}$ and w_t^{FS} share some implementation difficulties. They require *(i)* to measure Σ_t and μ_t and *(ii)* to invert Σ_t, and both steps complicate the implementation of these optimal portfolio allocations. They also have their own disadvantages. To obtain the allocation w_t^{FS}, we need to model the joint dynamics of μ_t and Σ_t to estimate the expected value $E\left[\mu_t'\left(\mu_t\mu_t' + \Sigma_t\right)^{-1}\mu_t\right]$. While the allocation $w_t^{Markowitz}$ does not depend on such an unconditional moment, it only maximizes the UMVC up to an approximation as stated above.

A third approach proposed by Brandt et al. (2009) (BSV) is to bypass steps *(i)* and *(ii)* and directly model portfolio allocations as a parametric function of instruments,

$$w_t^{BSV} = w_t^{Benchmark} + X_t\theta, \tag{5}$$

where $X_t = (x_{t,i}, ..., x_{t,N})'$ is a N-by-K matrix of K instruments for N assets and θ is a K-by-one vector of parameters to be estimated.[4] Instruments can include firm character-

[4]For simplicity, we consider the case in which the number of assets N is the same every period. Otherwise, BSV normalize the second term by N_t to ensure the weight deviations are not impacted by number of assets available each period.

istics, macroeconomic variables, and their interactions. The parameters θ determine how the allocation for asset i, $w_{t,i}^{BSV}$, deviates from its benchmark allocation, $w_{t,i}^{Benchmark}$, as a function of instruments $x_{t,i}$. The benchmark portfolio usually is the value- or equal-weighted portfolio that includes all N assets. Finally, the parameters θ are estimated by maximizing the sample estimator of the UMVC using past data.[5]

The BSV methodology is especially suited when the investment universe contains assets whose risk/return profiles depend on the same set of instruments and the cross-section N is large enough to estimate the parameters θ. In the next section, we propose a new methodology to model portfolio allocations that can be applied when the number of assets N is small or there are no common instruments available.

2.1 A parametric approach using historical returns

In the section, we propose our new methodology. Our objective is to develop a methodology that can be used with only past returns, can be applied in cases where the number of assets is small or large, and allows for allocation constraints.

In our approach, we explicitly account for allocation constraints such as no short-selling. To do so, we parameterize portfolio allocations as a function of a N-by-one vector of latent variables, f_t, as $w_t = W(f_t)$. For example, when short-sales are precluded we use the exponential function $W(f_t) = e^{f_t}$ and when the portfolio is further restricted to be fully invested we use the function $W(f_t) = e^{f_t} / \sum_{j=1}^{N} e^{f_{t,j}}$. In the case with no allocation constraints, we set $W(f_t) = f_t$. The no-short-sales and fully invested case is akin to a mutual fund whereas the unconstrained case is closer to a hedge fund. While we focus on these two cases in the main text, Appendix A contains other examples of portfolio weight constraints handled by our methodology.

[5] While we focus on a mean-variance criterion, the BSV approach can also be applied with other utility functions because the portfolio allocations are not derived from the investor's objective.

We model the latent variables f_t using an auto-regressive process as,

$$f_{t+1} = (1-\beta)\,\bar{f} + \beta f_t + \alpha s_t. \tag{6}$$

In this equation, the N-by-one vector $\bar{f}$ captures the long-run average values of f_t, the N-by-one vector s_t are lagged shocks to portfolio allocations, and α and β are scalar parameters. The parameter β is constrained to be between 0 and 1.

The main difficulty in using a time-series modeling approach for portfolio allocations is to find the appropriate shock s_t. The key point of our methodology is the characterization of s_t. We use the gradient of the investor's UMVC in Equation (1) with respect to f_t as,

$$\begin{aligned} s_t &= \frac{\partial \mathrm{UMVC}}{\partial f_t}, \\ &= \left(\frac{\partial w_t}{\partial f_t}\right)' \frac{\partial \mathrm{UMVC}}{\partial w_t}, \\ &= \frac{1}{T}\left(\frac{\partial w_t}{\partial f_t}\right)' \left(r_t - \gamma\left(r_{p,t} - \bar{r}_p\right) r_t\right), \end{aligned} \tag{7}$$

where we have used the chain rule to obtain the second equality, $\left(\frac{\partial w_t}{\partial f_t}\right)_{i,j} = \frac{\partial w_{t,i}}{\partial f_{t,j}}$ is a N-by-N matrix of partial derivatives, the sample estimator of the UMVC is used to obtain the last equality, and $\bar{r}_p = \frac{1}{T}\sum_{t=1}^{T} r_{p,t}$ is the average portfolio return.[6]

We can use the following intuition to see why Equation (7) is the appropriate allocation shock to use. The gradient of the UMVC with respect to f_t tells us in which direction to change the allocations at time t to improve the UMVC. Computing the gradient requires knowing the returns realized at time t and therefore, cannot be used at the beginning of period t when we allocate the portfolio. However, if the expected returns and covariance matrix exhibit persistence over time, then the time-t gradient can be used to improve the portfolio allocations for period $t+1$. The parameters α and β in the dynamic process for f_t in Equation (6) indicate the extent to which past gradients can be used to optimally change

[6]The optimal parameter values of α and β are such that we have $E[s_t] = 0$. Therefore, with our choice of dynamic in Equation (6) we obtain that the vector $\bar{f} = E[f_t]$ is the unconditional expected value of f_t.

the current allocation.

The form of the portfolio allocation shock s_t is intuitive. Consider first the case of an unconstrained investor with $w_t = f_t$. The matrix of partial derivative, $\frac{\partial w_t}{\partial f_t} = I_N$, is the identify matrix of size N and the shock to the allocation of asset i becomes

$$s_{t,i} \quad = \quad \frac{1}{T}\left(r_{t,i} - \gamma\left(r_{p,t} - \bar{r}_p\right) r_{t,i}\right), \quad i = 1, ..., N. \tag{8}$$

If the estimated parameter α is positive, then the first element, $r_{t,i}$, increases the allocation to asset i at time $t+1$ because high past returns are indicative of higher future expected returns. In contrast, high past values of $\left(r_{p,t} - \bar{r}_p\right) r_{t,i}$ will lower the allocation at time $t+1$, all else equal, because they signal a higher covariance between asset i and the investor's portfolio p. The tradeoff between these two effects is captured by the investor's risk aversion γ.

In cases where the allocation is constrained, the matrix of partial derivatives $\frac{\partial w_t}{\partial f_t}$ accounts for the transformation between the variables f_t and portfolio allocations w_t. In the empirical application, we consider both an unconstrained investor and an investor who cannot short-sell, $w_{t,i} \geq 0$, and is fully invested in risky asset, $\sum_{i=1}^{N} w_{t,i} = 1$. We use the function $W(f_t) = \frac{e^{f_t}}{\sum_{j=1}^{N} e^{f_{t,j}}}$ and obtain

$$\frac{\partial w_t}{\partial f_t} = diag(W(f_t)) - W(f_t)W(f_t)'. \tag{9}$$

To interpret the impact of the partial derivatives of allocations, consider the case with two assets. The shock to the allocation of the first asset is

$$s_{t,1} \quad = \quad \frac{1}{T} W_1(f_t)(1 - W_1(f_t))\left[\left(r_{t,1} - \gamma\left(r_{p,t} - \bar{r}_p\right) r_{t,1}\right) - \left(r_{t,2} - \gamma\left(r_{p,t} - \bar{r}_p\right) r_{t,2}\right)\right], \tag{10}$$

where we have used the fact that $W_2(f_t) = 1 - W_1(f_t)$ in the two asset case. Therefore, when the allocation to the first asset gets close to either 0 or 1, then $W_1(f_t)(1 - W_1(f_t))$

approaches 0 and the randomness in the first asset's allocation is shut down as $s_{t,1}$ is close to 0. In such case, $f_{t+1,1}$ reverts back up or down to $\bar{f}_1$ because of the term $\beta f_{t,1}$ in Equation (6). When the allocation to the first asset is not 0 or 1 and α is positive, then the allocation increases with the term $r_{t,1} - \gamma\left(r_{p,t} - \bar{r}_p\right) r_{t,1}$ as in the unconstrained case, but decreases with $r_{t,2} - \gamma\left(r_{p,t} - \bar{r}_p\right) r_{t,2}$ because positive values indicate that the second asset has become more attractive in mean-variance terms.

As in the BSV approach, the parameters $\bar{f}$, α, and β are estimated by numerically maximizing the UMVC over all past returns. The time-variations in portfolio allocations depend on the shocks s_t in Equation (7) which themselves depend on portfolio allocations through the average portfolio return $\bar{r}_p$. To operationalize this approach, each iteration of the numerical optimization goes through two steps. First, we compute portfolio returns using Equations (6) and (7) with $\bar{r}_p = 0$. Then, we rerun the computations using the average portfolio return from the first step.

Finally, in our empirical tests, we remove the factor $1/T$ to allow a comparison of the estimated α parameters across different sample sizes. Removing this term has no incidence on the methodology. We refer to our approach as auto-regressive portfolio allocations (ARPA). Next, we empirically evaluate its out-of-sample performance.

3 Empirical application

To empirically measure the benefit of ARPA, we conduct out-of-sample portfolio tests on different sets of assets. We start by describing the sets of assets and the benchmark portfolio allocations. Then, we detail how we implement the out-of-sample tests.

3.1 Data

We use weekly returns on different sets of assets that include equity portfolios, equity factors, government bonds, and commodities. Table 1 presents descriptive statistics for each set of assets. We report the start dates, the number of assets, the average across assets of the average returns, volatilities, and cross-correlations. Given that the ARPA methodology aims to benefit from persistence in return moments, we also compute the first-order autocorrelation coefficients of returns, absolute returns, and cross-products of returns. We average the first two measures across all assets and the last across all pairs of assets. The first measure captures the persistence in the means, the second is a measure of the persistence in variances, and the third measures persistence in covariances. All data are weekly, in USD, and end in November 2019.

The first six sets of assets are those of DGU. The other sets contain either other equity risk factors that have been proposed in the past decade or other asset classes. First, we consider a set that contains ten value-weighted portfolios of U.S. stocks sorted by major industries. The next set includes eight developed market equity indices: Canada, France, Germany, Italy, Japan, Switzerland, U.K., and U.S. The next set includes the U.S. equity factors in the Fama-French three-factor model (Fama and French, 1993); the value-weighted market portfolio (MKT), the small-minus-big market capitalization factor (SMB), and the high-minus-low book-to-market ratio factor (HML). The final sets of assets from DGU include different combinations of 25 market capitalization and book-to-market ratio sorted value-weighted U.S. stock portfolios with the previous three factors as well as the high-minus-low previous 12-month return (skipping month $t-1$) momentum factor (MOM, Jegadeesh and Titman, 1993; Carhart, 1997). As in DGU, we remove from the sets *25 Size/BM + MKT*, *25 Size/BM + FF3*, and *25 Size/BM + FF3 + MOM* the five portfolios with the largest market capitalizations to prevent cases where assets would be too highly correlated.

The set *30 Industries* contains 30 portfolios of U.S. stocks sorted by industry. We use this set to measure the performance of the ARPA methodology when N is large, but the portfolio sorting is not based on variables known to be related to expected returns like size and book-to-market ratio. Next, we consider other sets of equity risk factors beyond the Fama-French three-factor model. We first augment MKT, SMB, and HML with MOM.

Second, we use the Fama-French five-factor model with MKT, SMB, HML, a robust-minus-weak profitability factor (RMW), a conservative-minus-aggressive investment factor (CMA) that we augment with MOM. We also include a set *DM FF5* that contains MKT, SMB, RMW, CMA, and MOM for four regions (North America, Developed Europe, Japan, and Asia Pacific).

Finally, we use two sets of assets that contain multiple asset classes. First, we combine MKT and the 10-year U.S. government bond. Second, we use MKT, the 10-year U.S. government bond, the Datastream Developed Market ex North America index for international stocks, the total return on the S&P Goldman Sachs Commodity Index, and the price of gold. All U.S. equity data are obtained from Kenneth French's website, the government bond index from CRSP, and international stocks and commodity data from Datastream.

Our choice of sets of assets covers different investment styles. An investor may be interested in the *10 Industries* or *30 Industries* set to try to generate outperformance by rotating across defensive and aggressive industries during the business cycle. The different combinations of market capitalization and book-to-market ratio sorted portfolios are used by investors to benefit from the return predictability of these characteristics. The choice between using the 25 value-weighted portfolios or the long-short equity factors depends on the investor's ability to short-sell stocks. The set *U.S. Equity/Bond* considers the classic asset allocation problem of how to optimally split a portfolio between stock market investments and safer government bonds. Finally, the *International* and *Equity/Bond/Commodity* sets represent other potential investment opportunity sets for both individual and institutional investors.

U.S. equity data start in 1926 except for RMW and CMA, which start in 1963. The U.S. government bond data start in 1961, the international equity indexes start in 1973, and the developed market factor data start in 1990. Our sets cover both the case in which the number of assets is small, for example, two for *U.S. Equity/Bond* and three for *FF3*, and as large as 30 in the *30 Industries* set. Recall that the parametric portfolio rule methodology of

Brandt et al. (2009) is difficult to implement in the former cases when the number of assets is too small.

We next report the average returns and volatilities averaged across all assets in each set. Sets with equity factors or bonds have lower returns and volatilities than sets with equity portfolios. Portfolio diversification benefits are largely determined by correlations between assets. We report in the sixth column the cross-correlation averaged across all pairs of assets in each set. Cross-correlation for sets with equity factors and bonds range from -0.05 to 0.08. Given these low correlations, simple combinations of these assets are bound to generate substantial portfolio diversification benefits. In contrast, the other sets exhibit higher than 0.52 average correlations. In such cases, optimally combining these assets into portfolios is more challenging.

Average return autocorrelations are positive but small. None are higher than 0.08. In contrast, returns exhibit higher autocorrelations in absolute returns and return cross-products, ranging from 0.21 to 0.34 and from -0.03 to 0.25, respectively. These averages suggest that temporary deviations in portfolio allocations are more likely to be driven by time-variations in the covariance matrix than in expected returns.

Overall, we consider sets of assets that mainly vary according to the number of assets and their average cross-correlation. We next describe the two alternative portfolio allocations we use to benchmark the ARPA portfolios.

3.2 Benchmark allocations

We use two portfolio rules to benchmark the ARPA. First, we follow DGU and use the equal-weighted allocation as

$$w_{t,i}^{EW}(s,\gamma) = \frac{1}{N(s)} \quad \forall i \in \{1, ..., N(s)\}.$$

where s is the set of assets. Note that this allocation does not depend on the risk aversion γ and depends on s only through the number of assets $N(s)$.

Second, we use the sample mean-variance estimator of $w_t^{Markowitz}$ as

$$w_t^{SMV}(s,\gamma) = \frac{\hat{\Sigma}_t^{-1}\hat{\mu}_t}{\gamma}$$

where $\hat{\Sigma}_t$ and $\hat{\mu}_t$ are sample estimators of the covariance matrix and average excess returns, respectively, using all past data (i.e., using returns from time 1 to $t-1$). In the next section, we describe how we implement the out-of-sample portfolio allocation tests.

3.3 Out-of-sample tests

For each set of assets s and each risk aversion level γ, we determine a start date, $T_{Start}(s)$, which corresponds to a quarter of the sample size ($T_{Start}(s) = \lceil T(s)/4 \rceil$) or 10 years ($T_{Start}(s) = 10 \times 52$), whichever is the largest. Then, each month $t = T_{Start}(s), T_{Start}(s) + 1, ..., T(s)$ we follow these steps:

1. We estimate parameters using all past observations from the start of the sample period to $t-1$. We estimate Σ_t and μ_t to compute $w_t^{SMV}(s,\gamma)$ and $\bar{f}$, α, and β to obtain $w_t^{ARPA}(s,\gamma)$.[7] For the ARPA methodology, we re-estimate the parameters once a year. While the parameter estimates are updated annually, the allocations are updated weekly.

2. Following Ao et al. (2019), we compute the net of transaction cost performance for the different portfolio rules as

$$r_{p,t}^j(s,\gamma) = \left(1 - \sum_{i=1}^{N} c_{t,i}|w_{t,i}^j(s,\gamma) - w_{t-,i}^j(s,\gamma)|\right)\left(1 + r_{f,t} + \left(w_t^j(s,\gamma)\right)' r_t\right) - 1,$$

   where $j = EW, SMV, ARPA$ and $c_{t,i}$ is a cost level that measures the transaction cost

[7] We omit the dependence of Σ_t and μ_t on s and of $\bar{f}$, α, and β on s and γ to lighten the notation.

per dollar traded for trading asset i and $w^j_{t-,i}(s,\gamma)$ is the portfolio weight for asset i at the beginning of period t before rebalancing. For simplicity, we set $c_{t,i} = 0.5\%$ for all t and i. Transaction costs affect the realized performance of all strategies. But most importantly, they affect the estimation of the ARPA model. Indeed, the numerical optimization of the UMVC to estimate $\bar{f}$, α, and β takes into account transaction costs when computing the UMVC. For example, a higher α value is chosen only if it brings enough performance benefits to counteract the increased turnover.

At the end of November 2019, we compute the realized UMVCs as

$$UMVC^j(s,\gamma) = \hat{E}[r^j_{p,t}(s,\gamma)] - \frac{\gamma}{2}\widehat{Var}[r^j_{p,t}(s,\gamma)] \tag{11}$$

where the sample average return $\hat{E}$ and sample return variance $\widehat{Var}$ are computed using portfolio j's returns from $t = T_{Start}(s)$ to $t = T(s)$. Finally, we consider investors with different levels of risk aversion $\gamma \in \{1, 3, 5, 10\}$ and with different allocation constraints (unconstrained or with no short-selling and full-investment constraints). We present the out-of-sample results for the unconstrained investor in the next section and analyze contrained allocations in Section 3.7.

3.4 Out-of-sample performance for unconstrained portfolios

We first focus on unconstrained investors. This case is usually challenging because the Markowitz portfolio rule is known to maximize estimation errors, see Michaud (1989). Therefore, unconstrained optimized portfolio allocations typically result in large infeasible weights and poor out-of-sample performances.

Table 2 contains our main results. We report the realized unconditional mean-variance criteria in percent per year for the EW, SMV, and ARPA portfolios for all risk aversion levels. We bootstrap the realized time series of portfolio returns to assess the significance of the difference in UMVCs between the SMV and the EW portfolios and between the ARPA and the EW portfolios.[8]

First, the UMVCs across risk aversion levels for the SMV portfolios are more often than not lower than the UMVCs for the EW portfolio. Even worse, they are, in some cases, negative; the SMV portfolio destroys value on a risk-adjusted basis. These results echo the findings of DGU. They are more often negative for the most risk-seeking investor with $\gamma = 1$.

In sharp contrast, the ARPA portfolio outperforms the EW portfolio for all but two sets containing equity factors or size and book-to-market ratio sorted portfolio: *FF3*, *25 Size/BM*, *25 Size/BM + MKT*, *25 Size/BM + FF3*, *25 Size/BM + FF3 + MOM*, *FF4*, *FF5 + MOM*, *DM FF5*. The outperformances are statistically significant, except in a few cases when risk aversion is high. The outperformances for the set *DM FF5* are not significant, despite their economic magnitude, because of the short sample period.

The industry portfolios and international equity portfolios significantly outperform when risk aversion is high. The performance for multi-asset class sets is lower than for the EW portfolio. Finally, in all cases, when the SMV significantly outperforms the EW portfolio, we find that the ARPA portfolio further outperforms. This result indicates the benefit of the dynamic portfolio rebalancing modeled in the ARPA methodology.

On average across all sets of assets and risk aversion levels, the proportional increase in realized UMVCs going from the EW to the ARPA portfolio is more than 100% (unreported). However, there are important differences across sets of assets and risk aversion levels. Therefore, to gain a better perspective on these differences, we further explore the determinants of outperformance in the next section.

3.5 Under which conditions does ARPA outperform?

In this section, we run panel regressions to better understand under which conditions do ARPA portfolios outperform the EW benchmark. Precisely, we investigate the determinants of the contribution to performance for each week t, each asset set s, and each risk aversion

[8] We create 10,000 bootstrap samples. We also used the block bootstrap of Politis and Romano (1994) and find similar results.

γ defined as,

$$UMVC_t^{ARPA}(s,\gamma) = r_{p,t}^{ARPA}(s,\gamma) - \frac{\gamma}{2}\left(r_{p,t}^{ARPA}(s,\gamma) - \hat{E}[r_{p,t}^{ARPA}(s,\gamma)]\right)^2. \qquad (12)$$

In this equation, $r_{p,t}^{ARPA}(s,\gamma)$ is the return at time t for the ARPA portfolio for asset set s and risk aversion level γ. The realized mean-variance criteria, $UMVC^{ARPA}(s,\gamma)$, defined in Equation (11) and reported in Table 2 are the time-series averages of $UMVC_t^{ARPA}(s,\gamma)$.

We similarly compute the performance contributions, $UMVC_t^{EW}(s,\gamma)$, for the EW portfolios. Then, we compute the relative difference as

$$\Delta UMVC_t^{ARPA-EW}(s,\gamma) = \frac{UMVC_t^{ARPA}(s,\gamma) - UMVC_t^{EW}(s,\gamma)}{|UMVC^{EW}(s,\gamma)|}. \qquad (13)$$

In this equation, we compute the difference in performance contribution between the ARPA and EW portfolio at time t, and standardize it by the realized UMVC of the EW portfolio, $UMVC^{EW}(s,\gamma)$. We use the absolute value of the denominator to preserve the direction of the outperformance in the one case where the $UMVC^{EW}(s,\gamma)$ is negative.

We use the analysis of DGU to guide our empirical investigation of the determinants of ARPA's outperformance relative to EW. They analytically show that the sample-based optimal mean-variance allocation (i.e., the SMV portfolio) is expected to outperform the equal-weighted allocation if three conditions are met: *(i)* the estimation sample is long enough, *(ii)* the number of assets is small enough, and *(iii)* the *ex ante* squared Sharpe ratio of the optimized portfolio is substantially higher than the one for the EW portfolio. Condition *(i)* is explained by the need to have a large enough sample of returns to estimate return moments and condition *(ii)* implies that the number of return moments to estimate should not be too large relative to the sample size. Condition *(iii)* states that there should be a benefit for optimizing allocations relative to using an equally weighted allocation.

We run the following panel regression for the performance contribution of the ARPA portfolio relative to the EW portfolio on different explanatory variables,

$$\Delta UMVC_t^{ARPA-EW}(s,\gamma) = \kappa + X_{t,s,\gamma}\beta + \epsilon_{t,s,\gamma}, \tag{14}$$

where κ is a constant, $X_{t,s,\gamma}$ is a vector of explanatory variables, β is a vector of coefficients, and $\epsilon_{t,s,\gamma}$ are error terms. All regressors are demeaned and standardized by their respective range such that the estimated constant is the average relative outperformance across all asset sets and risk aversion levels. We combine observations for all weeks, asset sets, and risk aversion levels, resulting in $153,524$ observations. We cluster standard errors by asset set.

We first run a regression using only a constant. Column *(i)* in Table 3 contains the estimate and its t-ratio. The estimate of 1.14 is the average of all $\Delta UMVC_t^{ARPA-EW}(s,\gamma)$ values, meaning that the average realized mean-variance criterion of the ARPA portfolio is 114% higher than the one for the EW portfolio. This average is significant, with a t-ratio of 4.12.

Next, we add as an explanatory variable the risk aversion level γ. Column *(ii)* reports a significantly positive estimate. Capturing time-variations in expected returns remains the major difficulty in dynamic portfolio management. Therefore, portfolios for highly risk averse investors better perform because they put less emphasis on timing expected returns and take less risk.

Column *(iii)* contains estimates for a regression specification that tests whether the ARPA outperformance depends on the same three conditions identified by DGU for sample-based mean-variance allocations. We include as explanatory variables the estimation sample size at each point in time $t-1$, the number of assets $N(s)$, and a measure of the difference between the squared Sharpe ratios of the ARPA and EW portfolios. Each period t, we measure the maximum squared Sharpe ratio as $S^{Max,2} = \hat{\mu}_t'\hat{\Sigma}_t^{-1}\hat{\mu}_t$ where $\hat{\mu}_t$ and $\hat{\Sigma}_t$ are estimated using returns from the first time period to $t-1$. Similarly, we compute the squared Sharpe ratio of the EW portfolio $S^{EW,2}$ and compute the ratio $S^{Max,2}/S^{EW,2}$.

The outperformance is positively related to the length of the estimation sample, $t-1$. The ARPA portfolio, just as the SMV, performs better when a long estimation sample is available. Next, we find a positive coefficient for the number of assets, which is the opposite of what condition *(ii)* for the SMV portfolio implies. In that case, a low number of assets means that the number of expected returns, variances, and correlations to estimate, which amounts to $\frac{N(s)^2+3N(s)}{2}$ parameters, is small. In the case of the ARPA methodology, there are $N(s)+2$ parameters to estimate (α, β, and all the $\bar{f}_i$). Therefore, the number of parameters to estimate for the ARPA portfolio is linear in the number of assets, whereas it is quadratic for the SMV portfolio. Despite the larger number of $\bar{f}_i$ parameters to estimate, a large number of assets is beneficial for the ARPA methodology because of the larger cross-section of allocation shocks, s_t, available to estimate the parameters α and β governing the temporary allocation deviations.

Then, we find a negative coefficient for the ratio of squared Sharpe ratios, while condition *(iii)* says that this coefficient should be positive for the SMV portfolio. We consider the possibility that the difference in squared Sharpe ratios may be a noisy measure of the potential outperformance of the optimized portfolio. We add in specification *(iv)* the mean absolute weight difference at each point in time between the ARPA and the EW portfolios. We expect to find higher outperformance in sets during weeks when the difference in allocation between the two portfolios is larger.

As expected, the coefficient of 5.42 is positive and highly significant. Therefore, the closer is the ARPA to the EW portfolios, as in the case of the *Industry* and *U.S. Equity/Bond* sets, the smaller is the outperformance. This result is intuitive: if the optimal allocation is close to $1/N$ and we face estimating errors when estimating allocations, then we are better off adopting the equal-weighted allocation.

On the other hand, we find that for the sets of assets whose optimal allocations differ from the equal-weighted allocations, then the outperformance is large and significant. This result is not mechanical; we could have obtained different portfolio allocations and poorer performance, as in the case of the SMV portfolio.

The definition of the performance contribution $UMVC_t^{ARPA}(s, \gamma)$ in Equation (12) uses the sample average portfolio return computed using all returns in the contribution to realized portfolio variance. To ensure that this use of forward-looking information does not impact our results, we also run the panel regressions with $UMVC_t^{ARPA}(s, \gamma)$ computed with the average portfolio return up to time $t-1$. All the results are virtually unchanged.

Overall, we find that the ARPA methodology provides benefits in out-of-sample portfolio performances, especially when optimal allocations differ from the equal-weighted portfolio. In the next section, we examine the dynamics of the estimated ARPA parameters and allocations.

3.6 How do β and α vary?

To gain more intuition on the ARPA results, we examine in this section the dynamics of the estimated parameters and the dynamic portfolio allocations.

To save space, we focus our discussion on four representative sets of assets: *10 Industries*, *International*, *FF3*, and *Equity/Bond/Commodity*.[9] Figure 1 reports the time-series of estimated α and β parameters for all risk aversion levels. Recall that these parameters are estimated each week using all past returns.

The β parameters are all above 0.9 and they do not differ across risk aversion levels. These high values indicate that deviations from the strategic asset allocations are highly persistent.

Differently, the α parameters reported in the right graphs vary between sets of assets. They go up to 0.6 for *10 Industries*, up to 4 for *International* and *FF3*, and up to 15 for *Equity/Bond/Commodity*. The αs also consistently vary across risk aversion levels. The lower the risk aversion, the higher is the estimated α. However, these higher α values do not

[9] All results are available from the author.

necessarily imply that the risk tolerant investor trades more. Instead, they are indicative of the larger scales of the latent variables, f_t, and allocation shocks, s_t, for investors with low risk aversion.

To see this point more clearly, Figures 2 to 5 respectively report for each set of assets the dynamic allocations over time. First, we see in Figures 4 for the set *FF3* the sources of outperformance of the ARPA versus the EW portfolio. All investors, except the most risk averse one, take on leverage to generate outperformance. Further, they allocate consistently to the MKT and HML factors, and very little to the SMB factors, whereas the EW portfolio is constantly invested in the SMB factor with a 33.33% allocation.

All dynamic allocations show a small number of short positions and persistent allocations to some assets. Across all sets, we unsurprisingly find that portfolio leverage decreases as risk aversion increases.

In summary, we obtain better out-of-sample performance for unconstrained portfolios using the ARPA methodology. The generated outperformance is obtained by adopting persistent leveraged long and small short positions. In the next section, we consider the case of a constrained investor who cannot short sell and is fully invested in risky assets, constraints similar to those of a mutual fund.

3.7 Out-of-sample performance for constrained portfolios

We report in Table 4 the realized unconditional mean-variance criterion for the ARPA portfolios in which we constrain allocations to be non-negative and to sum to one. We compare their performance to the EW and constrained SMV portfolios. The constrained SMV portfolio is found by numerically optimizing the historical mean-variance criterion using constant non-negative allocations that sum to one.

We find that the ARPA portfolio outperforms the EW portfolio for the same sets of assets as for the unconstrained investors, but less so when risk aversion is high. Unsurprisingly, the magnitude of the outperformance is smaller for constrained allocations than for unconstrained allocations in Table 2.

The constrained SMV portfolio performs better than the unconstrained SMV portfolio, in line with the findings of Jagannathan and Ma (2003). In several cases, the ARPA does not outperform the constrained SMV portfolio. Therefore, the ARPA methodology is more effective at delivering outperformance when the portfolio allocations are not constrained.

To gain a better perspective on these results, we estimate the panel regression in Equation (14) using the performance contributions of the constrained portfolios as the left-hand-side variable. The results are presented in Table 5.

The average outperformance across all asset sets and risk aversion levels is 0.25 and is always significant at the 1% level. Therefore, even in the case with allocation constraints, the ARPA methodology significantly increases the realized UMVC compared to the EW portfolio. While the improvement was more than 100% in the unconstrained case, it is 25% in the constrained case.

We find as in the unconstrained case that outperformance is positively related to the risk aversion level, the length of the estimation sample, the number of assets, and the weekly average allocation difference between the ARPA and EW portfolios, although the last two coefficients are not significant.

In contrast to the unconstrained case, we find that outperformance of the constrained portfolio is positively related to the squared Sharpe ratio difference. This result is in line with DGU's condition *(iii)* for the outperformance of the SMV versus the EW portfolio.

Overall, the results in Table 5 are consistent with those in Table 3. The notable exception is that the unconstrained outperformance is significantly related to average allocation difference, whereas the constrained outperformance is significantly related to the ratio of squared Sharpe ratios.

Finally, Figures 6-9 present the allocations over time for the sets *10 Industries*, *International*, *FF3*, and *Equity/Bond/Commodity*.Allocations significantly vary over time and across risk aversion levels. Consider the set *FF3* for example, the investor with a low risk aversion ($\gamma = 1$) is almost always invested in the market portfolio to achieve higher average return and risk, whereas the investor with a high risk aversion ($\gamma = 10$) adopts a more balanced allocation across MKT, SMB, and HML.

4 Conclusion

We provide a new methodology to model dynamic mean-variance portfolio allocations to optimize an unconditional mean-variance criterion. We parameterize portfolio allocations using an auto-regressive process in which the shock is the gradient of the investor's realized mean-variance criterion with respect to portfolio allocation. Our methodology handles transaction costs, allocation constraints, and can be used even if there are no instruments available to model time-variations in portfolio allocations. Given these features, our methodology significantly contributes to the literature on parametric portfolio rules.

We run empirical tests using equity portfolios, equity long-short factors, government bonds, and commodities. We find that our methodology outperforms benchmark portfolio rules.

The time-series approach we use to model portfolio allocations makes it possible to extend our methodology in different directions. In Equation (6), α and β are scalar parameters. Hence, we restrict the persistence in f_t to be the same for all assets and preclude cross-asset effects in which the lagged shock for one asset could impact the current weight for another asset. We can generalize the auto-regressive process for portfolio allocations to use more general parameter matrices to account for asset-specific weight persistence and cross-asset effects.

Also, we can include a set of characteristics X_t on the right-hand side of the allocation dynamic to nest the BSV approach and empirically evaluate the value of our allocation shock compared to other instruments used in the literature. We leave these points for further research.

A Allocation constraints and associated gradients

We considered in our main results, an unconstrained investor and an investor constrained to have positive allocations and be fully invested in risky assets. In this appendix, we provide the portfolio allocation shocks for other allocation constraints: the case with only a no short-selling constraint and the case with only the constraint that allocations sum to one.

A.1 No short-sales

We use the function $W(f_t) = e^{f_t}$ to prevent short-sales. The portfolio allocation shocks are then

$$s_t = \frac{1}{T} diag(e^{f_t}) \left(r_t - \gamma \left(r_{p,t} - \bar{r}_p\right) r_t\right). \tag{15}$$

The effect of the adjustment e^{f_t} is the following. If $f_{t,i}$ is a small number, then $e^{f_{t,i}}$ is close to zero. In such case, there is no shock to the allocation of asset i and $f_{t+1,i}$ reverts back up to $\bar{f}_i$ because of the term $\beta f_{t,i}$ in Equation (6). Therefore, randomness is shut down when the allocation of an asset gets close to 0.

A.2 Fully invested in risky assets

We use the function $W(f_t) = \frac{f_t}{\sum_{j=1}^{N} f_{t,j}}$ to ensure weights sum to 1. The portfolio allocation shocks are then

$$s_t = \frac{1}{T} \left(\frac{I_N - W(f_t)\iota_N'}{\sum_{j=1}^{N} f_{t,j}} \right)' \left(r_t - \gamma \left(r_{p,t} - \bar{r}_p\right) r_t\right), \tag{16}$$

where ι_N is a N-by-ones vector of ones.

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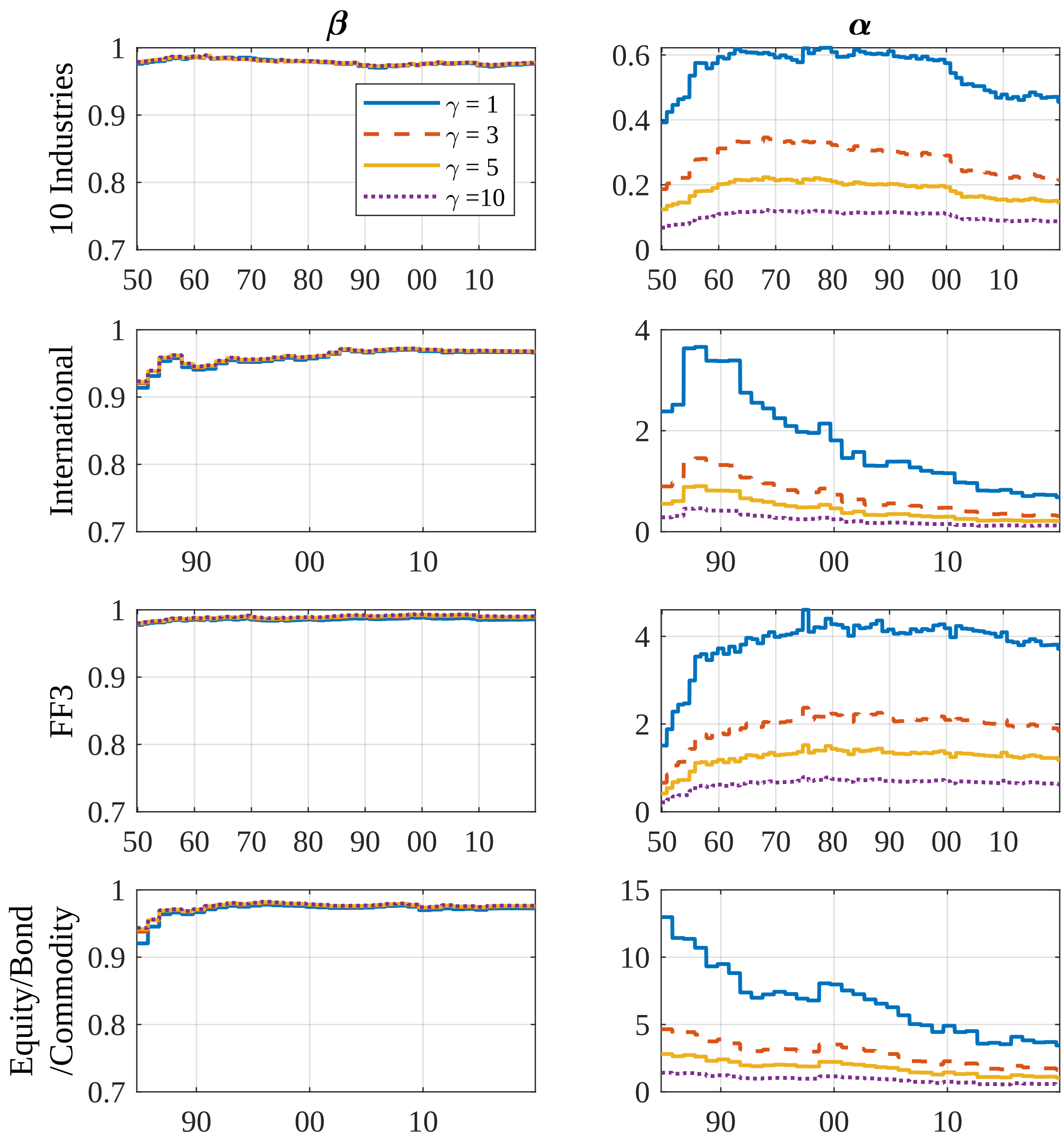

Figure 1 Estimated ARPA parameters
We report the estimated parameters β and α for the ARPA methodology for unconstrained investors with different risk aversion levels. We report on the set *10 Industries* in the first row, *International* in the second row, *FF3* in the third row, and *Equity/Bond/Commodity* in the fourth row. The left graphs contain estimated βs and the right graphs report on estimated αs. Each week, the parameters are estimated by maximizing the unconditional mean-variance criterion in Equation (1) on all past returns. All data are weekly, in USD, and end in November 2019.

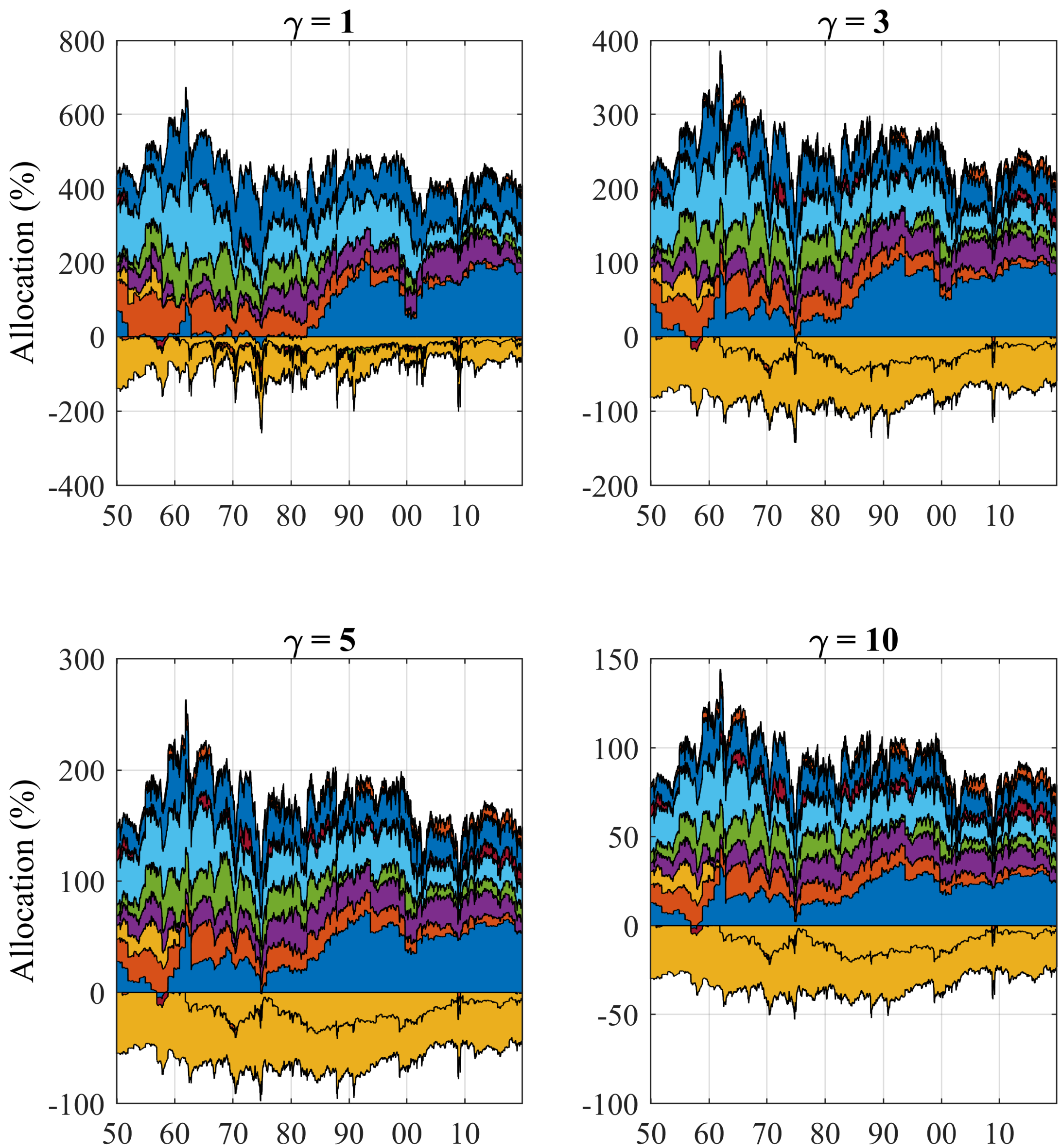

Figure 2 Dynamic unconstrained portfolio allocations - 10 Industries
We report the unconstrained investors' allocations, $w_t^{ARPA}(s, \gamma)$, for the set of assets that contains the 10 industry U.S. stock portfolios. Each panel reports on a different risk aversion level. All data are weekly, in USD, and end in November 2019.

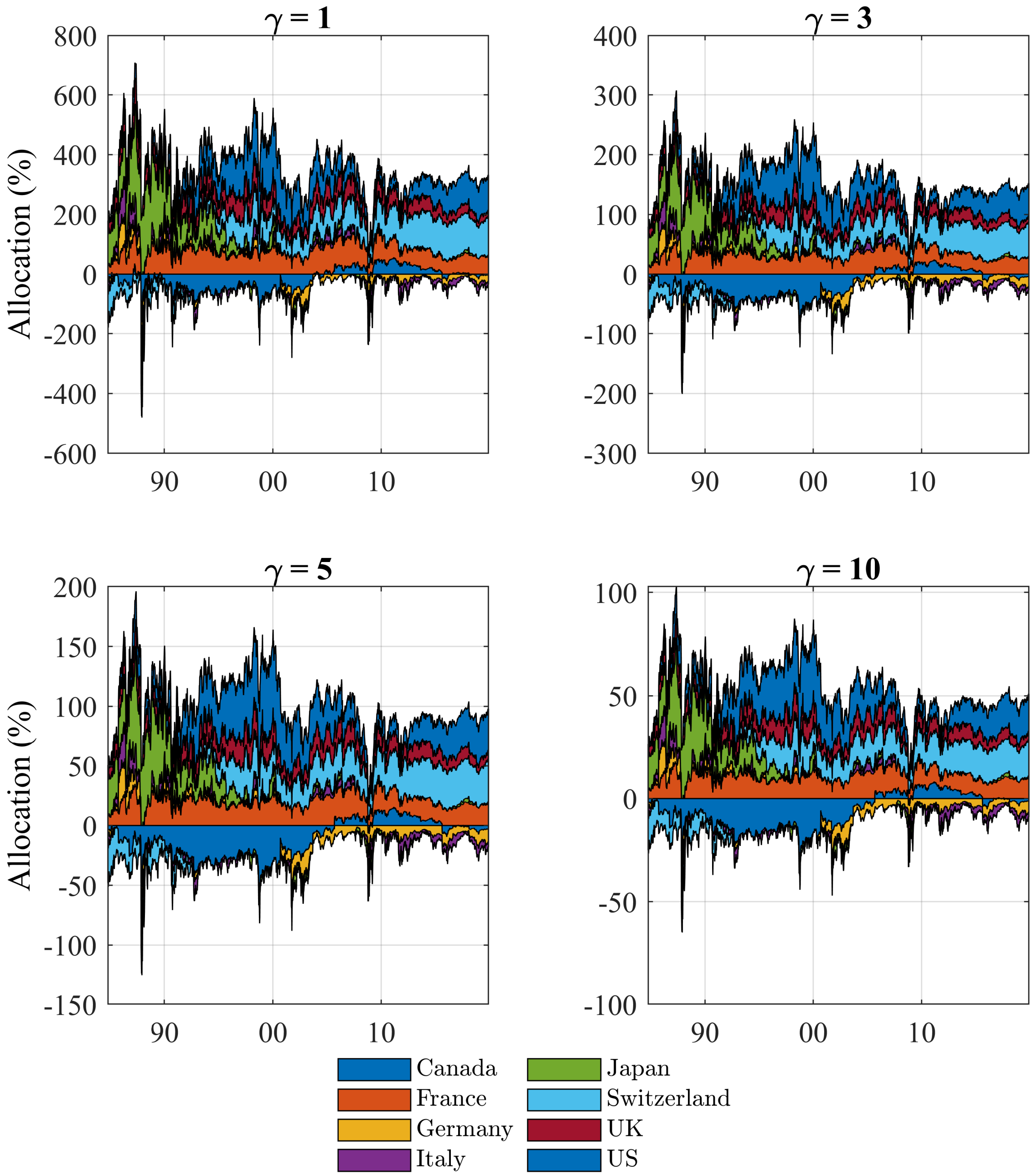

Figure 3 Dynamic unconstrained portfolio allocations - International
We report the unconstrained investors' allocations, $w_t^{ARPA}(s, \gamma)$, for the set of assets that contains the eight developed market equity indexes. Each panel reports on a different risk aversion level. All data are weekly, in USD, and end in November 2019.

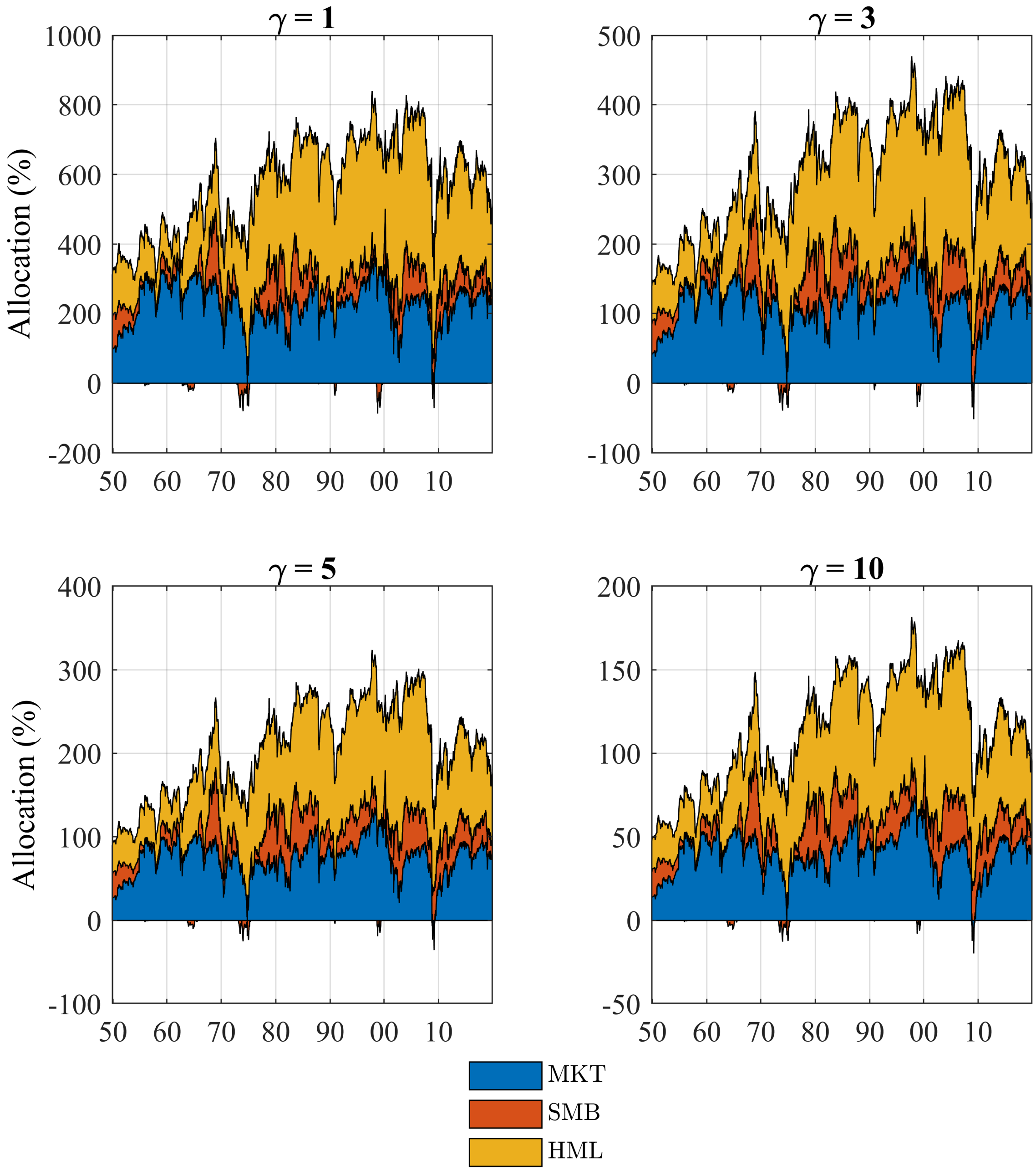

Figure 4 Dynamic unconstrained portfolio allocations - FF3
We report the unconstrained investors' allocations, $w_t^{ARPA}(s, \gamma)$, for the set of assets that contains the three Fama-French factors. Each panel reports on a different risk aversion level. All data are weekly, in USD, and end in November 2019.

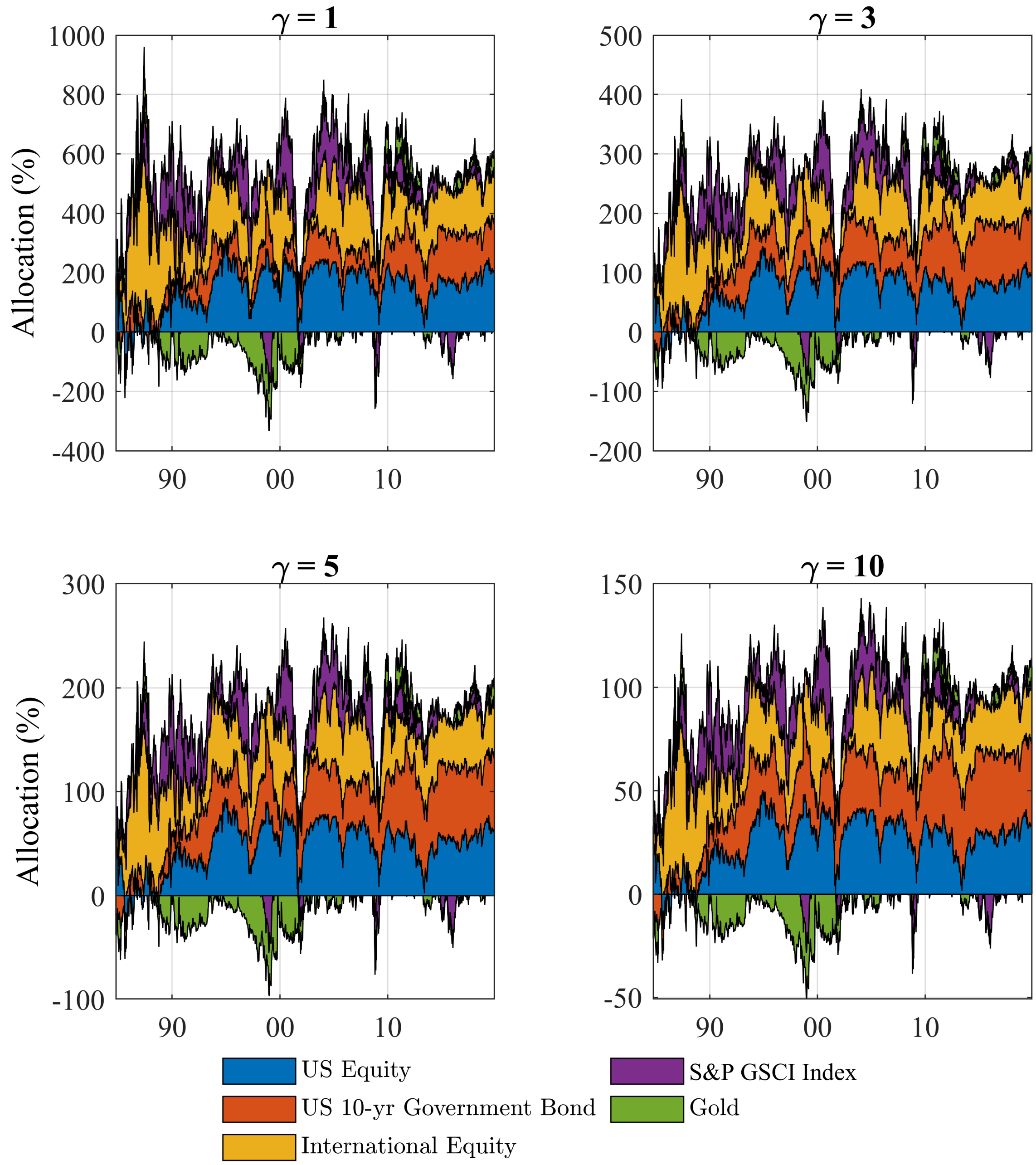

Figure 5 Dynamic unconstrained portfolio allocations - Equity/Bond/Commodity
We report the unconstrained investors' allocations, $w_t^{ARPA}(s, \gamma)$, for the set of assets that contains the U.S. stock market, the U.S. 10-year government bond, the developed market ex North America equity index, the S&P Goldman Sachs Commodity Index, and gold. Each panel reports on a different risk aversion level. All data are weekly, in USD, and end in November 2019.

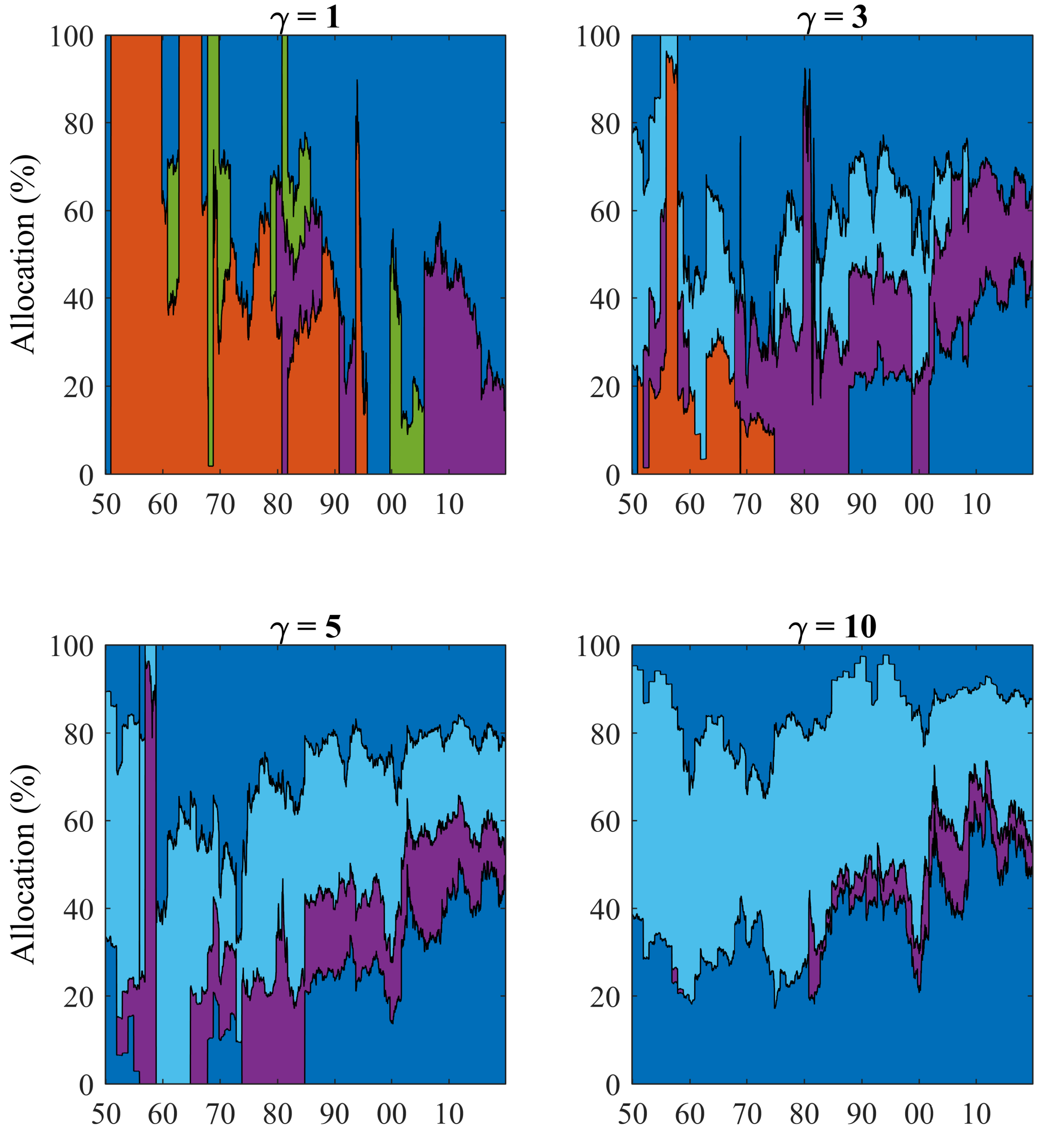

Figure 6 Dynamic constrained portfolio allocations - 10 Industries
We report the constrained investors' allocations, $w_t^{ARPA}(s, \gamma)$, for the set of assets that contains the 10 industry U.S. stock portfolios. Each panel reports on a different risk aversion level. All data are weekly, in USD, and end in November 2019.

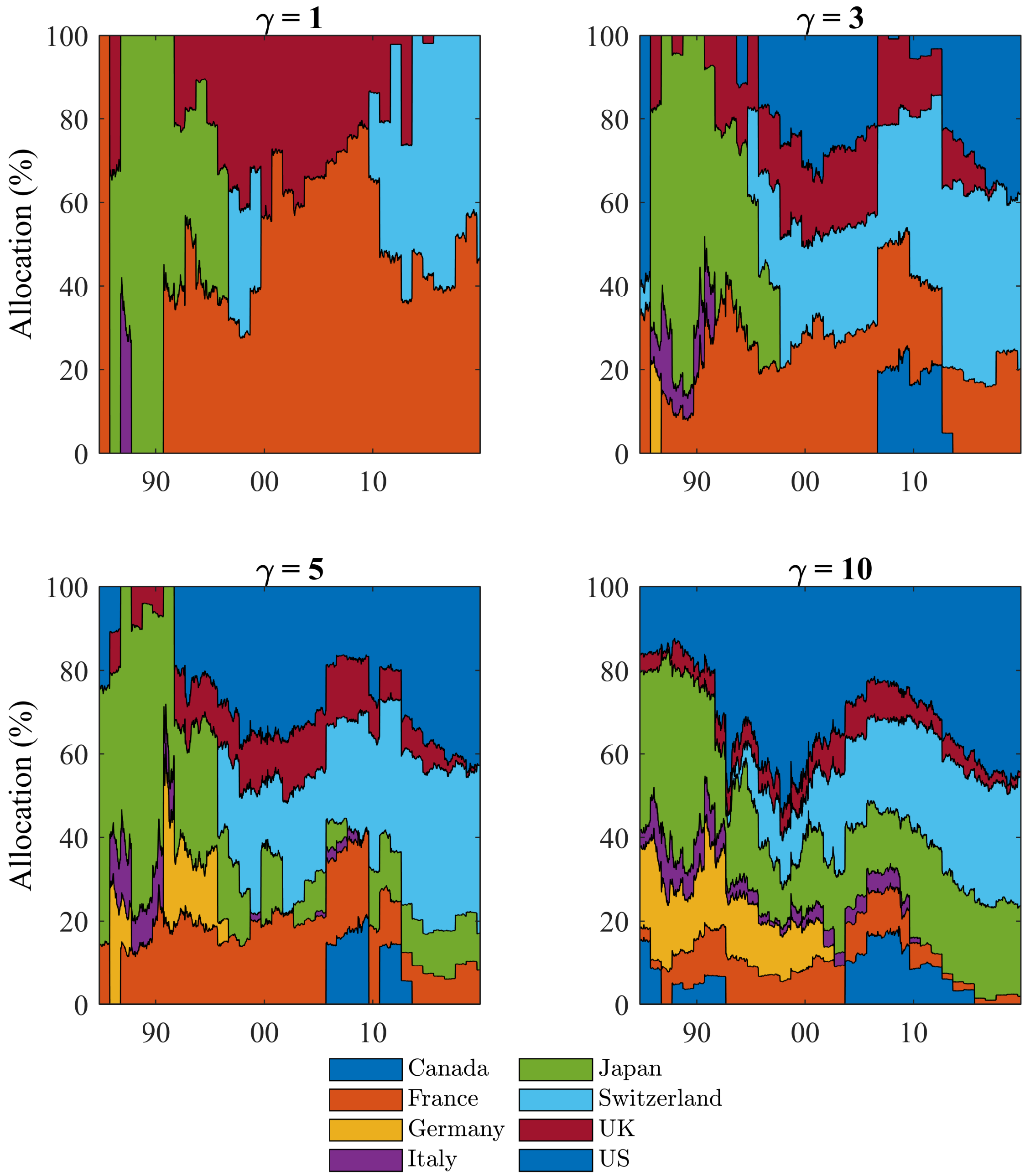

Figure 7 Dynamic constrained portfolio allocations - International
We report the constrained investors' allocations, $w_t^{ARPA}(s, \gamma)$, for the set of assets that contains the eight developed market equity indexes. Each panel reports on a different risk aversion level. All data are weekly, in USD, and end in November 2019.

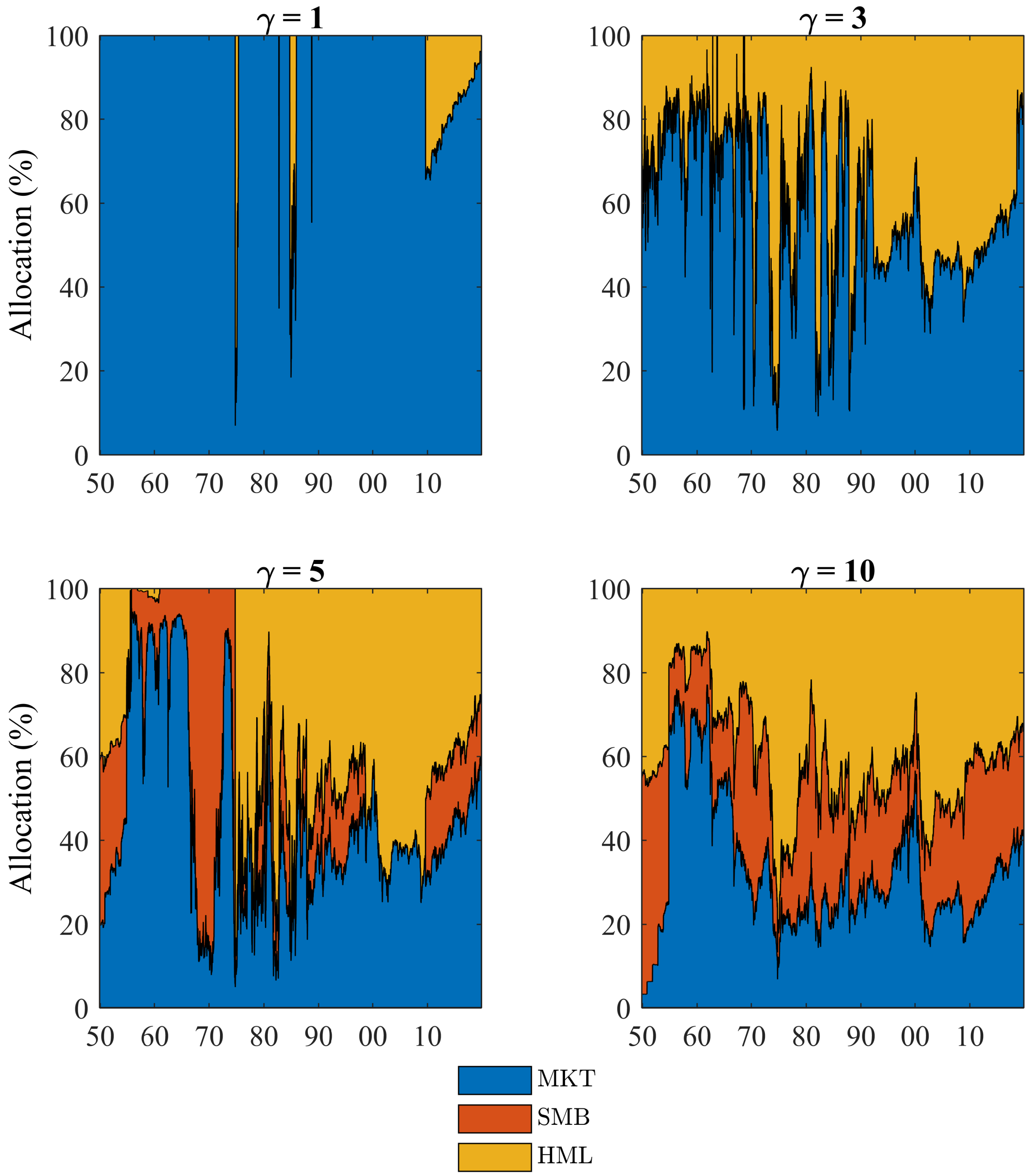

Figure 8 Dynamic constrained portfolio allocations - FF3
We report the constrained investors' allocations, $w_t^{ARPA}(s, \gamma)$, for the set of assets that contains the three Fama-French factors. Each panel reports on a different risk aversion level. All data are weekly, in USD, and end in November 2019.

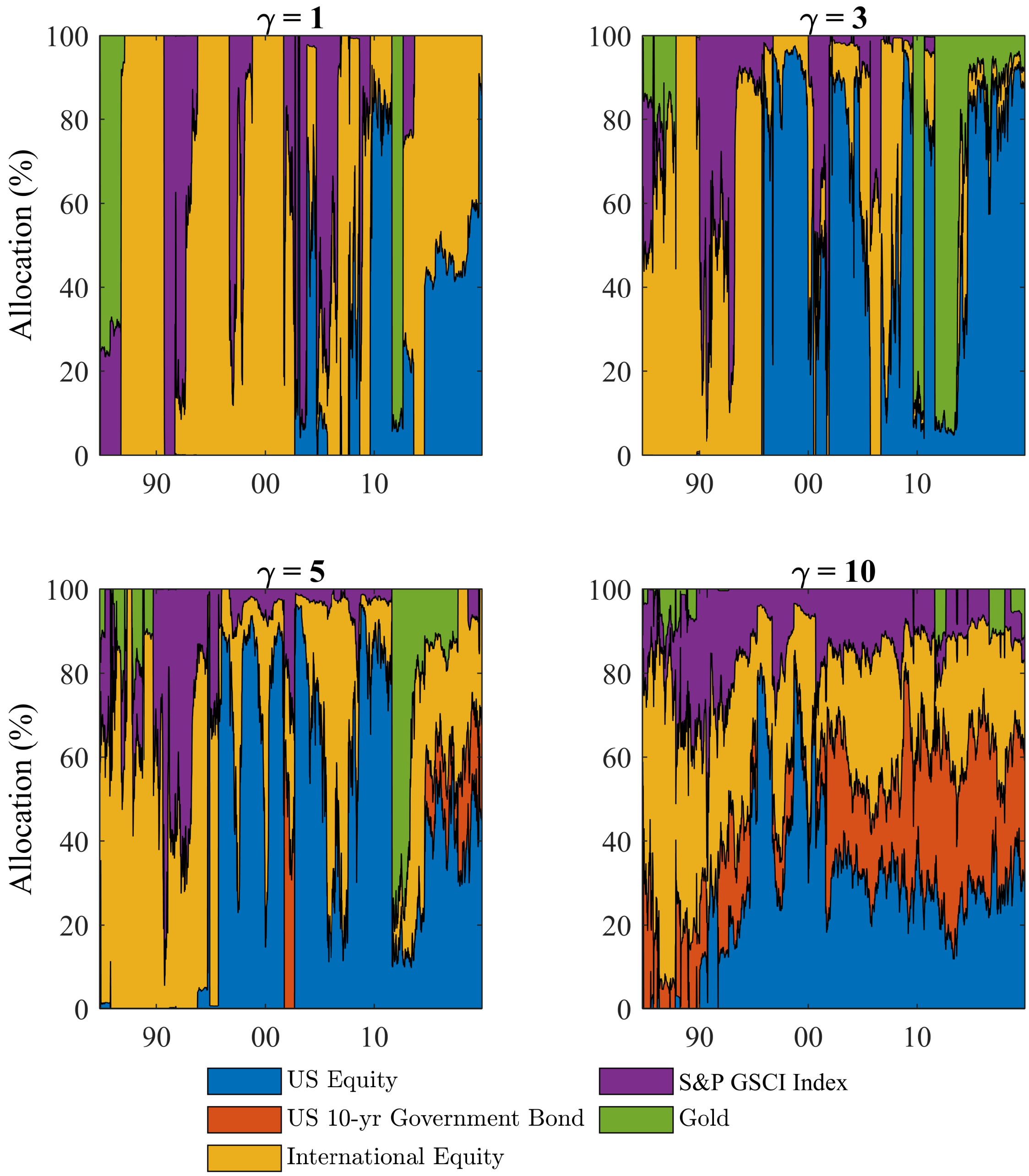

Figure 9 Dynamic constrained portfolio allocations - Equity/Bond/Commodity We report the constrained investors' allocations, $w_t^{ARPA}(s, \gamma)$, for the set of assets that contains the U.S. stock market, the U.S. 10-year government bond, the developed market ex North America equity index, the S&P Goldman Sachs Commodity Index, and gold. Each panel reports on a different risk aversion level. All data are weekly, in USD, and end in November 2019.

Table 1 Descriptive statistics

Set of assets	Start date	N	Average return	Average volatility	Average $Corr(r_{i,t}, r_{j,t})$	Average $Corr(r_{i,t}, r_{i,t-1})$	Average $Corr(\|r_{i,t}\|, \|r_{i,t-1}\|)$	Average $Corr(r_{i,t}r_{j,t}, r_{i,t-1}r_{j,t-1})$
10 Industries	Jul 26	10	11.61	19.86	0.71	0.02	0.29	0.25
International	Jan 73	8	11.09	19.75	0.52	−0.01	0.20	0.22
FF3	Jul 26	3	8.00	12.89	0.08	0.05	0.31	0.11
25 Size/BM	Jul 26	25	13.20	23.40	0.80	0.07	0.34	0.22
25 Size/BM + MKT	Jul 26	21	13.46	23.64	0.82	0.08	0.34	0.22
25 Size/BM + FF3	Jul 26	23	12.80	22.78	0.73	0.08	0.34	0.21
25 Size/BM + FF3 + MOM	Nov 26	24	12.75	22.50	0.65	0.08	0.34	0.21
30 Industries	Jul 26	30	11.77	23.03	0.62	0.02	0.28	0.22
FF4	Nov 26	4	8.70	13.15	−0.04	0.07	0.33	0.16
FF5 + MOM	Jul 63	6	9.26	10.44	−0.05	0.08	0.29	0.19
DM FF5	Nov 90	20	6.32	11.95	0.01	0.03	0.25	0.10
U.S. Equity/Bond	Jun 61	2	8.04	13.70	−0.00	0.03	0.22	−0.03
Equity/Bond/Commodity	Jan 73	5	8.62	15.97	0.05	0.02	0.21	0.03

We report summary statistics for the different sets of assets used in our empirical tests. All data are weekly, have different start dates reported in the second column, and end in November 2019. The average return is the time-series average return averaged across assets. The average volatility is the time-series return volatility averaged across assets. The average correlation is the time-series correlation between two assets averaged across all pairs of assets. We compute the first-order autocorrelation of returns and absolute returns and report their average values across assets. Finally, we compute the first-order autocorrelation of the cross-product of returns between two assets and report the average across all pairs of assets.

Table 2 Out-of-sample performance of unconstrained portfolios

	EW	SMV	ARPA	EW	SMV	ARPA
	$\gamma = 1$			$\gamma = 3$		
10 Industries	11.04	−8.06	11.61	9.08	5.00	7.37
International	10.22	−5.76	5.36	7.77	2.26	4.07
FF3	8.07	14.16*	17.33***	7.73	8.99*	10.69**
25 Size/BM	11.98	−18.29	23.00***	9.54	7.76	13.19**
25 Size/BM + MKT	12.12	−7.00	24.28***	9.56	8.56	13.52***
25 Size/BM + FF3	11.68	< -100.00	24.04***	9.51	−8.90	13.04**
25 Size/BM + FF3 + MOM	11.61	< -100.00	36.85***	9.63	−15.26	18.98***
30 Industries	10.94	−50.23	8.04	8.68	−2.26	5.34
FF4	9.11	22.19*	30.48***	8.88	16.89***	16.96***
FF5 + MOM	8.81	< -100.00	24.58*	8.67	−20.76	15.67
DM FF5	5.34	< -100.00	23.28	5.24	−56.09	12.67
U.S. Equity/Bond	8.53	3.61	7.22	7.89	4.97	5.80
Equity/Bond/Commodity	7.10	−0.02	6.32	6.50	5.18	5.39
	$\gamma = 5$			$\gamma = 10$		
10 Industries	7.13	5.11	6.30	2.24	4.78**	5.40***
International	5.31	2.88	3.85	−0.84	3.22**	3.68**
FF3	7.40	7.28	8.52	6.56	5.85	6.60
25 Size/BM	7.11	7.18	10.03**	1.03	5.90***	7.26***
25 Size/BM + MKT	6.99	7.40	10.13**	0.57	5.94***	7.28***
25 Size/BM + FF3	7.33	0.50	10.20**	1.91	3.47	7.41***
25 Size/BM + FF3 + MOM	7.65	1.06	13.76***	2.70	5.11	9.43***
30 Industries	6.42	1.30	4.89	0.78	3.00*	4.64***
FF4	8.65	12.54***	12.64***	8.07	8.64	8.67
FF5 + MOM	8.54	0.36	11.82	8.21	6.24	8.14
DM FF5	5.14	−11.99	8.23	4.88	1.59	4.40
U.S. Equity/Bond	7.24	4.94	5.41	5.62	4.85	5.08
Equity/Bond/Commodity	5.91	4.81	4.86	4.42	4.27	4.30

We report the realized unconditional mean-variance criteria, $UMVC^j(s, \gamma)$, (annualized in %) in Equation (11) for different portfolio allocations j, sets of assets s, and risk aversion levels γ. Columns EW report on a rule that rebalances each month to an equal-weighted allocation. Columns SMV report on the one-period Markowitz rule (see Equation 2) that uses each month the expected returns and covariance matrix estimated on all past returns. Columns ARPA report on the optimal dynamic allocations with parameters estimated each month using all past returns. We bootstrap the time-series of out-of-sample portfolio returns to obtain the significance of the difference between $UMVC^{SMV}$ and $UMVC^{EW}$ and between $UMVC^{ARPA}$ and $UMVC^{EW}$. We use 10,000 bootstrap samples. For each set, ***, **, and * denote significance at the 1%, 5%, and 10% level respectively. All data are weekly, in USD, and end in November 2019.

Table 3 What are the determinants of performance?

Regressor	*(i)*	*(ii)*	*(iii)*	*(iv)*
Constant	1.14^{***}	1.14^{***}	1.14^{***}	1.14^{***}
	(4.12)	(4.12)	(5.16)	(4.76)
γ		2.41^{**}	2.41^{**}	3.17^{***}
		(2.06)	(2.06)	(2.73)
$t-1$			1.41	1.62^{*}
			(1.45)	(1.69)
$S^{Max,2}/S^{EW,2}$			-0.63	-0.78
			(-0.72)	(-0.84)
$N(s)$			1.59^{**}	2.48^{***}
			(2.05)	(2.83)
$\frac{1}{N(s)}\sum_{i=1}^{N(s)} \lvert w_{t,i}^{ARPA}(s,\gamma) - w_{t,i}^{EW}(s,\gamma)\rvert$				5.42^{***}
				(4.52)

We run the panel regression (14) in which the time t performance contributions for the ARPA portfolio relative to the EW portfolio (see Equation (13)) are regressed against different explanatory variables. The explanatory variables include the risk aversion level, the size of the estimation sample, the ratio of the maximum squared Sharpe ratio to the equal-weighted allocation's squared Sharpe ratio, the number of assets, and the mean absolute difference in portfolio allocations between the ARPA and EW portfolios. All variables except γ and $N(s)$ are computed each week. The panel includes all weekly observations for all sets of assets and risk aversion levels, resulting in $153,524$ observations. Explanatory variables are demeaned and standardized by their respective range so that the estimate of the constant is the average proportional outperformance of the ARPA portfolio across all sets of assets and risk aversion levels. We report estimates with t-ratios in parentheses. We cluster standard errors by sets of assets, and ***, **, and * denote significance at the 1%, 5%, and 10% level respectively. All data are weekly, in USD, and end in November 2019.

Table 4 Out-of-sample performance of long-only and fully invested portfolios

	EW	SMV	ARPA	EW	SMV	ARPA
	$\gamma = 1$			$\gamma = 3$		
10 Industries	11.04	9.86	9.04	9.08	8.71	8.20
International	10.22	8.50	9.10	7.77	6.81	6.93
FF3	8.07	9.91**	9.53	7.73	8.17	8.32
25 Size/BM	11.98	14.76***	14.78***	9.54	10.47**	11.52***
25 Size/BM + MKT	12.12	14.76***	14.72***	9.56	10.63***	12.03***
25 Size/BM + FF3	11.68	14.76***	14.84***	9.51	10.14	9.44
25 Size/BM + FF3 + MOM	11.61	14.30***	14.53***	9.63	12.50***	11.52*
30 Industries	10.94	10.69	8.50	8.68	8.95	8.78
FF4	9.11	11.07**	10.65*	8.88	10.32**	10.47**
FF5 + MOM	8.81	11.18	11.22	8.67	9.59	9.53
DM FF5	5.34	5.62	9.27*	5.24	6.52	3.44
U.S. Equity/Bond	8.53	8.43	8.37	7.89	7.31	7.47
Equity/Bond/Commodity	7.10	4.62	3.42	6.50	5.98	5.29
	$\gamma = 5$			$\gamma = 10$		
10 Industries	7.13	7.31	6.81	2.24	3.36**	3.28*
International	5.31	4.87	4.38	−0.84	0.35*	0.27*
FF3	7.40	7.24	6.43	6.56	6.07	6.40
25 Size/BM	7.11	7.00	8.41**	1.03	1.10	0.75
25 Size/BM + MKT	6.99	7.03	8.81***	0.57	0.97	0.63
25 Size/BM + FF3	7.33	7.92	7.92	1.91	5.78***	6.01***
25 Size/BM + FF3 + MOM	7.65	11.46***	10.81**	2.70	8.98***	9.50***
30 Industries	6.42	7.19	6.83	0.78	3.10***	2.93***
FF4	8.65	9.60**	9.79**	8.07	8.41	8.61*
FF5 + MOM	8.54	8.44	8.34	8.21	7.77	7.41
DM FF5	5.14	6.62	3.18	4.88	5.79	4.04
U.S. Equity/Bond	7.24	6.62	6.63	5.62	5.20	5.14
Equity/Bond/Commodity	5.91	5.74	3.77	4.42	5.41*	4.63

We report the realized unconditional mean-variance criteria, $UMVC^j(s, \gamma)$, (annualized in %) in Equation (11) for different portfolio allocations j, sets of assets s, and risk aversion levels γ for a constrained investor who cannot short-sell and is restricted to be fully invested in risky assets. Columns EW report on a rule that rebalances each month to an equal-weighted allocation. Columns SMV report on the one-period Markowitz rule that maximizes the mean-variance criterion with static allocations. Columns ARPA report on the optimal dynamic allocations with parameters estimated each month using all past returns. We bootstrap the time-series of out-of-sample portfolio returns to obtain the significance of the difference between $UMVC^{SMV}$ and $UMVC^{EW}$ and between $UMVC^{ARPA}$ and $UMVC^{EW}$. We use 10,000 bootstrap samples. For each set, ***, **, and * denote significance at the 1%, 5%, and 10% level respectively. All data are weekly, in USD, and end in November 2019.

Table 5 What are the determinants of performance for constrained portfolios?

Regressor	*(i)*	*(ii)*	*(iii)*	*(iv)*
Constant	0.25^{***}	0.25^{***}	0.25^{***}	0.25^{***}
	(2.71)	(2.71)	(4.41)	(4.36)
γ		0.75^{**}	0.75^{**}	0.83^{**}
		(2.02)	(2.02)	(2.06)
$t-1$			1.17^{***}	1.19^{***}
			(3.08)	(3.18)
$S^{Max,2}/S^{EW,2}$			0.45^{*}	0.44^{*}
			(1.87)	(1.91)
$N(s)$			0.28	0.54
			(1.37)	(1.49)
$\frac{1}{N(s)}\sum_{i=1}^{N(s)}\lvert w_{t,i}^{ARPA}(s,\gamma)-w_{t,i}^{EW}(s,\gamma)\rvert$				0.66
				(1.27)

We run the panel regression (14) in which the time t performance contributions for the constrained ARPA portfolio relative to the EW portfolio (see Equation (13)) are regressed against different explanatory variables. The explanatory variables include the risk aversion level, the size of the estimation sample, the ratio of the maximum squared Sharpe ratio to the equal-weighted allocation's squared Sharpe ratio, the number of assets, and the mean absolute difference in portfolio allocations between the ARPA and EW portfolios. All variables except γ and $N(s)$ are computed each week. The panel includes all weekly observations for all sets of assets and risk aversion levels, resulting in $153,524$ observations. Explanatory variables are demeaned and standardized by their respective range so that the estimate of the constant is the average proportional outperformance of the ARPA portfolio across all sets of assets and risk aversion levels. We report estimates with t-ratios in parentheses. We cluster standard errors by sets of assets, and ***, **, and * denote significance at the 1%, 5%, and 10% level respectively. All data are weekly, in USD, and end in November 2019.